THE

ELEMENTS

OF

CURVES:

COMPRISING,

I.

THE GEOMETRICAL PRINCIPLES

OF THE

CONIC SECTIONS.

II.

AN INTRODUCTION

TO THE

ALGEBRAIC THEORY

OF

CURVES.

DESIGNED FOR

THE USE OF STUDENTS IN THE UNIVERSITY.

OXFORD,

AT THE CLARENDON PRESS.

MDCCCXXVIII.

B.g

A

SHORT TREATISE

ON THE

GEOMETRICAL PRINCIPLES

OF THE

CONIC SECTIONS.

PREFACE.

THE general complaints against existing systems of Conic Sections, whether as being in some instances too tedious and prolix, or in others not sufficiently systematic, elementary, or geometrical, have principally occasioned the present attempt to obviate them. The same consideration has also led to a much more hurried publication than the author would, under other circumstances, have ventured upon. But with those who know the state of mathematical studies in the University of Oxford at the present time, and how much their advancement calls for every assistance which can be given, and the removal of every unnecessary difficulty, an attempt like the present will, it is hoped, find some apology for its defects in the circumstances which have called it forth.

As to the method adopted, it is that of deducing the curves in the first instance from the cone: a method which has been supposed long and complex, not, as appears to the author, from any thing in the nature of the principle itself, but rather from the particular form in which it has in some instances been applied. By carrying this deduction only as far as one or two of the primary cases, from which all the other properties very readily follow in plano, it is hoped that this method will be found so simplified as to recommend itself, not less in brevity than in directness. The several descriptions of the curves in plano are afterwards given, and the properties which belong to them, as considered in each point of view, deduced upon purely geometrical principles.

As to the extent of the Treatise, the author trusts it will be found to contain all the principal or most useful properties of these curves, with reference to what must, in the present state of mathematical science, be the main objects of a geometrical system; in the first place, to afford a preparative for the study of Newton's Principia; and in the second, to form a suitable introduction to the general theory of curves investigated by the higher analysis;—objects which, notwithstanding the preference given by the continental writers, and of late by some in our own country, to the *exclusive* use of algebraic processes, seem still generally recognised in our academical courses as possessing many advantages, especially when these studies are regarded in reference to their use as an exercise of the intellectual faculties.

It may be necessary, for the sake of junior students, to make a remark or two respecting the mode of reasoning and notation here employed.

The reasoning in Part I. is offered as strictly geometrical: that is, dependent solely upon geometrical definitions and the geometrical properties of figures; no deduction being made by virtue of any algebraic operation upon symbols, even when representing geometrical quantities.

The mode of notation adopted might at first sight be thought to be an introduction of algebraic processes; but it is not so in fact: and the reader is desired carefully to bear in mind that these symbols are adopted only for the sake of brevity and perspicuity: the signs + and − never mean any thing more than the adding or taking away a given line or space. The sign of multiplication always means the rectangle contained under two given lines; and two quantities placed as a fraction simply express their ratio: thus $\frac{A}{B}$ is merely equivalent to $A : B$, and $\frac{A}{B}=\frac{C}{D}$ to $A : B :: C : D$.

This mode of expressing proportionals in many cases offers considerable advantages in point of brevity and clearness, from bringing at once under the eye all the steps of a proof, which if written at length would appear long and complex. Its utility will be perceived, and any apparent obscurity in it removed, upon the perusal of a very few of the propositions where it is exemplified.

An apology may possibly be also necessary for the introduction of a somewhat unusual mode of designating the different lines and quantities: but this has been done with the idea of rendering the investigations more clear to the learner by the use of a set of characters regulated upon one system throughout. The several symbols and contractions which are made use of, will hardly require any formal explanation: thus *VMU* signifies the rectangle contained by the straight lines *VM*, *MU*; the symbol "by ⊙" means, "by the property of the "circle;" a second line, or set of lines, analogous to a former, are represented by the same characters, distinguished by 2 subjoined, thus, *PM*, P_2M_2, &c.

In the second Part, the reasoning involves some departure from what are usually termed geometrical or elementary methods; that is to say, the doctrine of limits is introduced: and in some instances the deductions are made upon algebraic or trigonometrical principles. It is presumed that the student has become acquainted with the method of ultimate ratios before entering upon this second Part.

An Appendix is subjoined, containing a series of demonstrations which establish the principal propositions without any deduction from the cone, except the simple cases where the axis and its vertical tangents are concerned: these may be substituted for the others by those readers who prefer such a method.

CONTENTS.

PART I.

GEOMETRICAL AND ELEMENTARY PROPERTIES OF THE CONIC SECTIONS.

PART II.

HIGHER PROPERTIES.

APPENDIX.

CONIC SECTIONS.

ERRATA.

Page.	Line.	Error.	Correction.
4	17	PM_2^2	$P_2M_2^2$
23	8	we have,	we have by parallels,
31	8	PM_2	PM^2
32	14	B_2	B^2
35	3	EF^2	EF_2^2
36	9	$\frac{2}{1}$	$\frac{1}{2}$

Passim. The sign $<$ (less than) has been in many places confounded with $\angle$ (angle); as also $\triangle$ (triangle) with Δ. But these errors will hardly mislead the reader.

ADDENDA TO THE CONIC SECTIONS.

Pag. 5. After Art. 5. by Euc. II. 5.

$$VMU = (CV + CM)\,(CV - CM)$$
$$= CV^2 - CM^2.$$

In the hyperbola we must take $CM - CV$, or we have

$$VMU = CM^2 - CV^2.$$

This value is frequently used in the subsequent propositions.

P. 7. After Art. 13. add:

It is evident that if, instead of the secant T_2PQ, we had any tangent to the circle meeting PT_2 in T_2, the square of this tangent might be substituted in the proportion for the rectangle of the segments of the secant. This is the case referred to in Art. 28.

P. 30. After Art. (22.) add;

Hence, and from (15.)

$$\left.\begin{array}{l}\text{Ell. } PE + EF_2 + PF \\ \text{Hyp. } PE + EF_2 - PF\end{array}\right\} = 2A = 2PE$$

$\therefore$ Adding or subtracting equals.

$$\left.\begin{array}{l}\text{Ell. } PE - EF_2 \\ \text{Hyp. } EF_2 - PE\end{array}\right\} = PF$$

This is the property referred to in the proof of Art. 38, where obviously

$$\left.\begin{array}{l}\text{Ell. } PE^2-EF_2^2\\ \text{Hyp. } EF_2^2-PE^2\end{array}\right\}=\left\{\begin{array}{l}(PE+EF_2)\,(PE-EF_2)\\ (PF+EF_2)\,(EF_2-PE)\end{array}\right.$$

which by the above $= FPF_2$.

P. 32. to Art. (30) add;

$$F\tau^2=VF\,.\,FP\,.\,\frac{FU}{F_2P}.$$

The student will compare this expression, as well as those for the subtangent, normal, subnormal, &c. with those in the parabola, where the centre is infinitely remote.

PART I.

COMPRISING THE GEOMETRICAL AND ELEMENTARY PROPERTIES OF THE CONIC SECTIONS.

SECT. I.

THE CURVES DEDUCED FROM THE CONE.

On the Cone and its sections in general.

DEFINITION. A *right cone* is the solid generated by the revolution of a right angled triangle about one of its perpendicular sides. The base of the triangle consequently describes a *circle*, which is called the *base of the cone*. The hypothenuse of the triangle at any position of its revolution is called a *side* of the cone: the perpendicular about which it revolves, the *axis* of the cone; and its extremity the *apex*.

A plane cutting the cone and passing through the apex is called a *vertical plane*, or vertical section: it evidently forms a triangle. Any section of the cone by a plane *parallel to the base*, is evidently a *circle*.

If the axis be produced above the apex, and a similar triangle, in an inverted position with respect to the former, revolve about it in the same manner, it will generate a similar cone, inverted with respect to the other; which is called the *opposite cone*. (See Fig. 1.)

If the cone or opposite cones be cut by a plane not passing through the apex, nor parallel to the base, its intersection with the conical surface will neither be a circle nor a rectilineal figure; but will form a certain *curve*, the nature of which will be different according to the *inclination* of the plane: these curves are what are usually comprised under the name of the Conic Sections. Such sections are represented in Fig. (1.) where the cutting plane is taken in three different positions: in one, parallel to the opposite side of the cone; in a second, inclined at a less angle to the base; in a third, at a greater angle, and meeting the opposite cone where it forms an inverted curve.

In the first case, where the cutting plane is parallel to a side of the cone, the curve is called *a parabola;* its plane being parallel to the opposite side of the cone, the curve evidently never meets that side of the cone, but extends on each side as far as the cone itself is extended.

In the second case, the cutting plane is inclined to the base at a *less* angle than the side of the cone; and consequently cuts the cone through both its opposite sides. In this case the curve is called an *Ellipse:* it surrounds the cone, or is a curve returning into itself.

In the third case, the cutting plane is inclined to the base at a *greater* angle than the side of the cone forms with it; whence it will evidently not meet the opposite side of the cone: but if an opposite cone be formed, it will meet the opposite cone, and there form an inverted curve of the same kind: each of these curves extends towards the base of the respective cones as far as the cones themselves are extended. These curves are termed the *hyperbola* and the *opposite hyperbola*, or, named together, opposite hyperbolas.

We next proceed to define certain terms which belong to all these curves in common.

DEF. 1. If the axis of the cone be supposed drawn, and a plane passed through it at right angles with the plane of the section, the intersection of these two planes will give a straight line in the plane of the curve called *the axis* of the curve.

2. The point where the axis meets the curve is called a *vertex*. The parabola has evidently only one vertex; each opposite hyperbola one; and the ellipse two.

3. In the ellipse and hyperbola, the point of bisection of the axis between the vertices is called the *centre*.

4. A line perpendicular to the axis, terminated both ways by the curve, is called an *ordinate* to the axis. But this term is more commonly applied to *half the ordinate,* the whole being distinguished as "a double "ordinate."

5. The segment of the axis between the vertex and an ordinate is called an *abscissa*.

§. 1.

ELEMENTARY AND GENERAL PROPERTIES REFERRING TO THE AXIS, AND BELONGING TO ALL THE CONIC SECTIONS.

(1.) Prop. The axis bisects its ordinates.

(Fig. I.) The curve, its axis, and ordinates being constructed as by the definition, the intersection of the vertical plane through the axis of the cone, with a circle parallel to the base of the cone, is the diameter of the circle:

And the intersection of that circle with the plane of the curve is a perpendicular to its axis; (Euc. II. 19.) therefore (by the property of the circle) it is bisected by the axis; and by Def. it is an ordinate.

(2.) Prop. Case 1. The *rectangles of the abscissæ* are as the *squares of the ordinates* in the ellipse and hyperbola.

The same figure and construction remaining, we have by similar triangles the following proportions:

$$\frac{VM}{VM_2}=\frac{MB}{M_2\beta}\text{ and }\frac{UM}{UM_2}=\frac{AM}{\alpha M_2}$$

whence, by compounding the proportions, and substituting terms which are equal in value,

$$\frac{VMU}{VM_2U}=\frac{AMB=(\text{by }\odot)\ PM^2}{\alpha M_2\beta=(\text{by }\odot)\ P_2M_2^2}$$

(3.) Case 2. In the parabola, since there is no second vertex U, and from the parallelism $AM=\alpha M_2$, this proportion becomes

$$\frac{VM}{VM_2}=\frac{PM^2}{P_2M_2^2}$$

Or in the parabola the *abscissæ* are as the *squares of the ordinates.*

(4.) Def. In the parabola let L be a third proportional to a given abscissa VM and its ordinate; or, $VM.L=PM^2$: then from the above proportion we have

$$\frac{VM.L\,(=)\ PM^2}{VM_2.L\,(\therefore=)\,PM_2^2}$$

or the same quantity L is a *third proportional* to *any abscissa* and its corresponding *ordinate.* Since the ordinate, beginning from the vertex where it is nothing, has all possible values, there is some *double ordinate equal to L:* this is called the *latus rectum*, or *parameter to the axis.*

(5.) Def. In the ellipse, the ordinate to the axis which passes through the centre is called the *conjugate axis.* Writing A for the half axis, and B for the half conjugate axis, we have from (2)

$$\frac{A^2}{B^2}=\frac{VMU}{PM^2}$$

(6.) COR. It appears, from the formation of the curve in the cone, that the conjugate axis can never be greater than the first axis; it may be equal to it; in which case the section becomes parallel to the base, or the ellipse becomes a circle:

and we have $VMU=PM^2$.

(7.) DEF. The *latus rectum* is that double ordinate to the first axis, which is a third proportional to the *two axes.*

$$\text{Hence } \frac{A^2}{B^2}=\frac{A}{\frac{1}{2}L}.$$

(8.) DEF. In the hyperbola, the *Conjugate axis* is a *perpendicular* to the axis through the centre; determined in *length by a proportion*, the same as that just stated for the ellipse.

The *latus rectum* is similarly defined.

(9) COR. It appears, from the formation of the curves in the cone, that the conjugate axis may be either equal to, greater or less than, the first axis.

If the axes be *equal*, the hyperbolas are called *equilateral*; and we have

$$A=B=\tfrac{1}{2}L$$
$$\text{and } VMU=PM^2.$$

(10.) DEF. Since the axes of the hyperbola may have any ratio to each other, it is obvious that there may be a pair of opposite hyperbolas whose axes shall be *equal* to those of a given pair, but having the ratio *inverted*. Such hyperbolas being described upon the *conjugate* axis of the former pair as their *first* axis, and having the *first* axis of the former as their *conjugate*, are called *conjugate hyperbolas*.

(12.) Def. Conceive a plane touching a cone along one of its sides: the intersection of such a plane with the plane of a conic section passing through that side of the cone is called *a tangent* to the conic section.

(12.) Prop. The tangent at the vertex of a conic section is perpendicular to the axis.

For conceive a plane touching the cone along the side which passes through the vertex; the vertical section through the axis of the curve will be perpendicular to this touching plane, and the plane of the curve will again be perpendicular to the vertical section; consequently their intersection will be perpendicular to the vertical section, and therefore perpendicular to the axis of the curve. (See Fig. 1, 2.)

Cor. A secant or line cutting a conic section, if it be perpendicular to the axis, is also a secant to a circle parallel to the base, which it meets in the same points.

(13.) Prop. Let the tangent to the vertex of a conic section meet another tangent to any point; which also produced meets a secant parallel to the vertical tangent: then *the squares of the 3 tangents, and the rectangle under the segments of the secant, are proportionals.*

For (Fig. 2.) conceive VPP_2Q to be points in any conic section; pass a side of the cone through P; the plane through, OP, TT_2 will touch the cone, and give parallel tangents πT $\pi_2 T_2$ to the circles parallel to the base. At T_2 let the tangent meet the secant parallel to the vertical tangent, cutting the curve in P_2Q: then we have

$$\frac{VT^2 = (\text{by } \odot)\ T\pi^2}{TP^2} = (\text{sim. } \triangle\text{s}) \frac{T_2\pi_2^2 = (\text{by } \odot)\ PT_2Q_2}{T_2P_2^2}$$

(14.) Prop. On the same axis with the same vertex let two conic sections of the same kind be described, as in Fig. (3.) where the two curves may represent either two parabolas, two hyperbolas, or portions of two ellipses :

Let ordinates to the two curves be drawn through any the same points in the axis, as PM πM, P_2M_2 π_2M_2.

Then the *ordinates through the same point are in a given ratio.* For (omitting in the parabola MU and M_2U) we have in any of the curves (by 2 and 3)

$$\frac{VM\,.\,MU}{VM_2\,.\,M_2U} = \frac{PM^2}{P_2M_2^2} = \frac{\pi M^2}{\pi_2M_2^2}$$

$\therefore \frac{PM}{\pi M}$ = a constant ratio = $\frac{B}{\beta}$ (ellipse and hyperbola).

If the second ellipse be a circle, or the second hyperbola equilateral, $\beta = A$ and the ratio becomes $= \frac{B}{A}$.

(15.) Cor. 1. Supposing the second ellipse to be a circle, join $C\pi$ and draw PR parallel to it, meeting the axis in Q. Therefore we have

$$\frac{\pi M^2 = VMU}{PM^2} = (\text{sim. } \triangle\text{s}) \frac{C\pi^2 = A^2}{PQ^2 \therefore = B^2}$$

Hence $QR = A \mp B$; or if QR = difference or sum of the semi-axes be placed in any position with its extremities in the lines VC CB at right angles, and QP be made equal to B, *the point P will be in the ellipse:* which may thus be described in plano.

(16.) **Cor. 2.** With any of the conic sections Let tPT be a tangent to the first curve at P. Through T and π draw $T\pi\tau$: then we have,

$$\frac{PM}{\pi M} = (\text{sim. } \triangle\text{s})\ \frac{tM_2}{\tau M_2} \underset{(\therefore >)}{\overset{(>)}{=}} \frac{P_2M_2}{\pi_2M_2}$$

or any such point τ lies above π_2, or $\tau\pi T$ is a tangent. Hence tangents from the extremities of the two ordinates meet the axis in the same point.

(17.) **Cor. 3.** If instead of taking $\pi\pi_2$ in a second curve, as above, we suppose them points in the same curve on the other side of the axis, the same demonstration will apply to shew that tangents at the opposite extremities of an ordinate will meet the axis in the same point.

(18.) **Cor. 4.** If again πM, instead of being the continuation of the ordinate PM, were (in the ellipse, and hyperbola) an ordinate at an equal distance from the opposite vertex, and still on the opposite side of the axis to PM, the tangent πT would meet the axis at an equal distance MT, and would be *parallel* to PT; and thence equal to it.

---

§. 2.

PROPERTIES DEDUCED FROM THE CONE, AND REFERRING TO ANY DIAMETERS.

PARABOLA.

(19.) **Lemma.** Pass planes through two given parallel lines, and a point not in the same plane: these planes intersect in a line parallel to each of the given lines.

(Fig. 4.) Let TP T_2P_2 be the given parallels. Pass planes through them and O. Taking any points TP in one line, and equidistant points T_2P_2 in the other, join TT_2 TO T_2O: and at PP_2 draw PH P_2H respectively parallel to TO T_2O, in the same planes: then the bases of the triangles $TT_2 = PP_2 < T_2 = < P_2$ $< T = <$ P: hence the △s are equal; $TO = PH$ $T_2O = P_2H$: and ∴ by parallels, OH is parallel to TP T_2P_2.

(20.) Lemma. (Fig. 5.) $T\tau$ is parallel to the tangent at H, to the circle $A\alpha H$. Take any chord $A\alpha$, join AH αH and produce to meet the parallel in $T\tau$: then since the < formed by the tangent with $\alpha H = H\tau T$ = < in alternate segments, αAH, we have similar △s

$$\alpha AH \ \tau HT,$$

$$\text{whence } \frac{AH}{\alpha H} = \frac{\tau H}{TH}.$$

$$\text{and consequently } AHT = \alpha H\tau.$$

(21.) Prop. Let two lines parallel to the axis cut the curve and meet a tangent: *the squares of the segments of the tangent are as the segments of the secants* between the curve and the tangent.

(Fig. 6.) Conceive PP_2V to be points in the parabola. The secants parallel to the axis TP T_2P_2 meet the tangent to V in TT_2; pass planes through TP T_2P_2 and O; these intersect in the side of the cone OH, parallel to the plane of the parabola, and to each of the secants, (by 19:)

These planes intersect the plane parallel to the base through T in AH αH: and that through T_2 in A_2H_2: also the plane of the parabola intersects the first plane in $T\tau$; then by sim. △s we have,

$$\frac{OH}{TP} = \frac{AH.\,TH}{AT.\,TH.}$$

and $$\frac{OH_2}{TP_2} = \frac{AH_2\,TH_2}{AT_2.\,TH_2}$$

but $$\frac{OH}{OH_2} = \frac{\alpha H\,.\,\tau H = (\text{by } 20)\; AH\,.\,TH}{AH_2\,.\,TH_2}$$

$$\therefore \frac{TP}{TP_2} = \frac{AT.\,TH}{AT_2\,TH_2} = (\text{by } \odot)\; \frac{T\pi^2}{T\pi_2^2} = \frac{TV^2}{TV_2^2} \text{ (as in 13.)}$$

Def. A *Diameter* is any line parallel to the axis, cutting the curve.

An *Ordinate* is a line cutting a diameter, terminated by the curve, and parallel to the tangent at the vertex of the diameter.

An *Abscissa* is the segment of a diameter between the vertex and an ordinate.

The *Parameter to any Diameter* is a third proportional to the ordinate and abscissa of that Diameter.

(22.) Cor. 1. Any diameter bisects its ordinates.

In (21) if $TP = TP_2$ then $TV = TV_2$; also PP_2 is parallel to TT_2 and $\therefore$ is an ordinate by Def. and is bisected by the diameter through V.

(23.) Cor. 2. *The abscissæ are as the squares of the ordinates.*

(Fig. 7.) Retaining the same construction; by equals and parallels.

$$\frac{VM}{VM_2} = \frac{PM^2}{P_2M_3^2}$$

Whence writing P for the parameter

by definition $VM \cdot P = PM^2$

also $TP \cdot P = VT^2$.

(24.) DEF. A *subtangent* is the part of any diameter intercepted between the points where it is met by an ordinate, and by a tangent at the extremity of that ordinate.

COR. (3.) *The subtangent=twice the abscissa.*

A tangent to P meets the diameter produced in τ, and a line parallel to the diameter through P_2 in τ_2; from (21) we have

$$\frac{\tau_2 P^2}{\tau_2 P_2 = (\text{by } \|\text{s}) \; 2\tau M} \; (=) \; = \; (\because) \; \frac{4 \cdot \tau P^2}{4 \cdot \tau V}$$

$$\therefore \tau M = 2\,\tau V = 2VM.$$

(25.) COR. Hence tangents at P and P_2, the opposite extremities of the ordinate meet the diameter produced, in the same point.

The axis is evidently a diameter, and the property (3) a particular case of the above: these latter properties also apply to the axis.

PROPERTIES DEDUCED FROM THE CONE REFERRING TO ANY DIAMETERS.

ELLIPSE AND HYPERBOLA.

DEF. Any straight line passing through the centre and terminated each way by the curve is called *a diameter*.

The ordinate and abscissa are defined as in the parabola.

(26.) Prop. Any diameter is *bisected* in the centre.

(Fig. 10, and 11.) If ordinates to the *axis* PM KM be drawn at equal distances on each side of the centre and on opposite sides of the axis, and their extremities PK joined with C, we shall have equal triangles PMC KMC. And from the right angles and the equality of the sides they are also equiangular: $\therefore < PCM = KCM$, or PC CK lie in one line forming a diameter: and since these segments are equal, it is bisected in the centre.

(27.) Cor. The *tangents* at the vertices of any diameter are *parallel.*

For by (18) the tangents at the extremities of equal ordinates to the axis on opposite sides are parallel, and by the above the junction of the points of contact is a diameter.

(28.) Prop. The *vertical tangents* to any diameter *are as the segments of any third tangent* which they respectively intercept.

Case 1. In the ellipse Fig. (8) VU are the vertices at which the parallel tangents VT UT_2 are drawn, meeting the tangent TPT_2 to any point P.

Through the parallel tangents, and O the apex, pass planes touching the cone, which will intersect in OH parallel to each of the tangents by (19.)

They also intersect planes parallel to the base through TT_2 in aH, TAH, $T_2A_2H_2$: which are tangents to the circles.

Then by similar $\triangle$s we have,

$$\frac{TA}{TV} = \frac{AH = (\text{by} \odot)\ \alpha H}{OH} = \frac{A_2H_2}{OH_2} = \frac{T_2A_2}{T_2U}.$$

Whence as in (13) we have

$$\frac{TV}{T_2U} = \frac{TA = T\pi}{T_2A_2 = T_2\pi_2} = \frac{TP}{T_2P}.$$

Case 2. In the hyperbola (Fig. 9.) TP a tangent at any point P in one curve, meets the parallel tangents to the opposite vertices U,V, in T,T_2.

Then the plane through TP and O will touch the cones along $PAOA$, and TA T_2A_2 will be tangents to the circles parallel to the base.

Planes through VT UT_2 and O will intersect in HOH_2 parallel to each (19): therefore producing πT π_2T_2 to meet OH, the △ $OH\pi$ is similar to πTV, and $OH_2\pi_2$ to π_2T_2U.

The plane $OH_2\pi_2$ also touches the opposite cone in $O\pi_3$, and forms a tangent $\pi_3H_2 = \pi_2H_2$; and since $O\pi_3 = O\pi_2$, the △ $O\pi_2H_2$ is equal and similar to △ $O\pi_3H_2$, therefore △ πTV is similar to π_2T_2U. Hence we have

$$\frac{TP}{T_2P} = \frac{TA = T\pi}{T_2A_2 = T_2\pi_2} = \frac{TV}{T_2U}$$

(29.) Cor. In the hyperbola since TP (see also Fig. 11) is necessarily less than PT_2, TV is also less than T_2U, and consequently if the tangent meet the diameter VU in τ, $V\tau < U\tau$, or τ, *lies between the centre and the curve* to which the tangent belongs.

(30.) *Scholium.*

By a construction very similar to that used in the above proposition we might deduce an analogous property, if instead of the lines VT UT_2 being tangents, they had been secants; so that we should have had two points of section corresponding to each point of contact. By joining each of such points with the apex, and passing planes as before, we should have a set of similar triangles on each side of the cone; from which, by composition of ratios, we might deduce the property, that the squares of the segments of the tangent TPT_2 are as the rectangles of the segments of the parallel secants which they meet. By a further modification of the came construction we might shew, that a similar property would hold good, if the tangents, instead of lying in the same line, were parallel at opposite parts of the curve. And further, it might be shewn in a way very similar, that if for these tangents again secants were substituted, we should have the rectangles of their segments proportionals. This is in fact one of the most general properties of the Conic Sections; and from it the subsequent properties would be immediately deduced. But the investigation being somewhat complex, we shall in the present treatise advance no further with these properties deduced from the cone than to the case of four tangents above investigated. By the help of this property alone, those of diameters in general may be readily established in the ellipse, as will presently appear; though doubtless with some sacrifice of generality and symmetry in the deduction: and we shall also find it necessary to depart still more widely from uniformity, in the corresponding case in the hyperbola.

Those readers who are desirous of further investigating these general properties are referred to Dr. Robertson's Conic Sections, Book I.

(31.) PROP. Let any tangent meet a diameter produced, and from the contact draw an ordinate: then the *semi-diam.*^{2}=the *rectangle of the segments* from the *centre* to the *ordinate*, and to the *tangent*.

Take the vertical tangents meeting in T and T_2, the tangent to any point P, which also meets the diameter produced in τ; then we have (Figs. 10 and 11.)

$$\text{(by 28.)}\ \frac{VT}{UT_2} = \frac{PT}{PT_2} = \text{(by } \|\text{s)}\ \frac{VM}{UM};$$

$$\text{also}\quad \ldots\ldots = \frac{\tau V}{\tau U}\ (\text{sim. } \triangle\text{s}).$$

Hence by proportionals, (and since (29) $\tau V < \tau U$ the lower sign belongs to the hyperbola,)

$$\frac{\frac{1}{2}(\tau U \pm \tau V) = C\tau}{\frac{1}{2}(\tau U \mp \tau V) = CV} = \frac{\frac{1}{2}(UM \pm VM) = CV}{\frac{1}{2}(UM \mp VM) = CM}$$

DEF. The *subtangent* is defined as in the parabola.

(32.) COR. By the proposition:

$$\begin{aligned} CV^2 &= CM.(CM \pm M\tau) \\ &= CM^2 \pm CM\tau \end{aligned}$$

$$\therefore\ CM\tau = \left\{ \begin{array}{l} CV^2 - CM^2\ (\text{ell.}) \\ CM^2 - CV^2\ (\text{hyp.}) \end{array} \right\} = VMU$$

$$\text{Whence we have } \tau M = VM.\frac{MU}{CM}$$

Or the subtangent =

$$\text{abscissa} \times \frac{\text{2d abscissa}}{\text{segment from centre to ordinate.}}$$

(33.) PROP. In the ellipse, any *diameter bisects its ordinates*.

In (Fig. 10.) let VU $B\beta$ be two diameters, such that each is parallel to the ordinates of the other: then (PM being the portion of the ordinate between the curve and the diameter) by the parallels we have $PM = CM$: and by (31), and proportionals

$$\frac{PM^2 = CM^2}{CB^2} = \frac{PM^2}{C\tau^2} = (\text{sim. } \triangle)\frac{\tau M . MC = (32) VMU}{C\tau . MC = (31) CV^2}$$

If the segment of the same ordinate on the other side of the diameter be taken, since the conjugate diameter is bisected in the centre, we shall have the same proportion with all the terms the same, except QM instead of PM, which $\therefore = PM$, or the ordinate is bisected by the diameter. Thus the equation *applies* to *PM as the half ordinate*, and we have the

(34.) Prop. *The rectangles* of the *abscissæ* are as the *squares* of the *ordinates.*

(35.) Prop. In the hyperbola, *any diameter bisects* its *ordinates.*

(Fig. 20.) If the vertical tangents to the conjugate axes be produced to meet each other, they form a rectangle whose diagonals pass through the centre. Let these diagonals be drawn and produced indefinitely.

1st. Let the vertical tangent meet one diagonal in L, and any ordinate to the axis PQ produced meet them in R, ρ; then by parallels and (8) we have

$$\frac{CV^2}{CB^2 = VL} = \frac{CM^2}{RM^2} = \frac{CM^2 - CV^2}{RM^2 - VL^2 \therefore = PM^2}$$

$$\therefore VL^2 = RM^2 - PM^2 = PRQ.$$

But from the bisection of the ordinate to the axis and of $R\rho$, $PR = Q\rho$

Whence $VL^2 = RP\rho$.

And this being true of any ordinate, let one through D meet the diagonals in H, N, $\therefore RP\rho = HDN$.

2dly, Let a tangent at D meet the diagonals in $T\tau$; draw the parallel PQ_2 which is $\therefore$ an ordinate to CD, then by similar triangles,

$$\frac{RP}{R_2P} = \frac{HD}{DT} \text{ and } \frac{P\rho}{P_2\rho} = \frac{DN}{D\tau}$$

$$\therefore \frac{RP\rho}{R_2P\rho_2} \frac{(\;=)}{(\therefore=)} \frac{HDN}{TD\tau}$$

and similarly $\rho_2 Q_2 R_2 = \tau DT$.

hence (Euc. II. 1. &c.) $R_2P = \rho_2 Q_2$, and $\therefore DT = D\tau$.

Hence the diameter bisects PQ_2.

(36.) Cor. (Fig. 20.) From the last article we have

$$R_2P\rho_2 = P R_2 Q_2 = R_2 M_2^2 - PM^2 = DT_2^2$$

Hence by parallels,

$$\frac{R_2M_2^2 - DT^2 = PM_2^2}{CM_2^2 - CD^2 = DM_2K} = \frac{DT^2}{DC^2}$$

And since the same is true with any other ordinate, we have in hyperbola as in the ellipse;

The *rectangles* of *the abscissæ* are as *the squares* of the *ordinates*.

(37.) *Scholium.* It is evident that the axes are diameters; and the above proposition respecting diameters and their ordinates includes as a particular case the property of the axes at first deduced. (2).

In like manner we have the following definitions:

DEF. The ordinate to any diameter of an ellipse which passes through the centre is called the *conjugate diameter*; and we have from the last proposition, (writing the first semi-diameter$=D$, and the semi-conjugate$=G$)

$$\frac{VMU}{PM^2}=\frac{D^2}{G^2}.$$

In the hyperbola, the *conjugate diameter* is defined as a line through the centre parallel to the ordinates of the first diameter, and *determined in length* by the above *proportion*.

DEF. The *parameter* is a third proportional to a diameter and its conjugate: hence (writing P for the parameter) the above ratio$=\frac{D}{\frac{1}{2}P}$.

(38.) COR. Hence in the hyperbola (Fig. 20.) $G=DT$; and completing the parallelogram CT, its diagonal GD is bisected by the diagonal CT.

Also since $D\tau=CG$, GD is also parallel to the other diagonal $C\tau$:

The whole parallelogram thus formed is said to be *circumscribed* about the conjugate diameters; and all such parallelograms have the *same diagonals* as that circumscribed about the *axes*.

(39.) In the ellipse, since the conjugate diameter is limited by the curve, all the properties of diameters apply to it. In the hyperbola, since it is not thus limited, we must have recourse to further considerations to deduce its properties.

DEF. A line drawn from any *point in the hyperbola to a conjugate diameter, parallel to the first diameter*, is called a *conjugate ordinate*.

PROP. If a conjugate ordinate be drawn, the *square of* the *segment* of the conjugate diameter between the centre and the conjugate ordinate, + the square of the *semi-conjugate diameter* is to the square of the *conjugate ordinate*, as the squares of the *conjugate* and *first diameters*. For from (37) we have, (Fig. 11.)

$$\frac{D^2}{G^2}=\frac{CM^2-D^2=PM_2^2-D^2}{PM^2=CM_2^2}=\frac{PM_2^2[-D^2+D^2]}{CM_2^2+G^2}$$

(40.) PROP. Let a tangent meet the conjugate diameter, and a conjugate ordinate be drawn from the point of contact; then the *semi-conjugate diameter*$)^2$ = the rectangle of the *segments* from the *centre* to the *tangent*, and to the *conjugate ordinate*.

For from (31) inversely, we have

$$\frac{CV^2}{CM^2}=\frac{C\tau}{CM};\text{ whence,}$$

$$\frac{CV^2}{CM^2-CV^2}=\frac{C\tau}{CM-C\tau=\tau M}=(\text{sim }\triangle\text{s})\frac{C\tau_2}{PM}$$

But the first ratio also $=\frac{G^2}{PM^2}$

Therefore $G^2=(PM=)\,CM_2\,.\,C\tau_2$.

(41.) Schol. These properties applied to the case of the *axes* lead us to observe, that here we have two distinct sets of properties referring to the *conjugate axis;* one belonging to ordinates drawn to it from the first curve; another belonging to those drawn to it from the second, or conjugate hyperbola.

With respect to *diameters* in general, it is obvious, that if any conjugate diameter meet the conjugate curves, ordinates may in like manner be drawn to it from the conjugate curves, and the proportion between the ordinates and abscissæ will hold good; but nothing has yet appeared to shew that the vertices of that diameter, as determined by the Def. will lie in the conjugate curves, and that consequently the first diameter becomes its conjugate: this, however, will be shewn subsequently.

(42.) Cor. In the ellipse and hyperbola, if one *diameter* be *parallel* to the *ordinates* of a second, the *second* is *parallel* to the *ordinates* of the first.

(Fig. 10, and 11.) Let CB be parallel to PM an ordinate to CV; join QC and produce till it meets the curve in D. Hence $QC=CD$, also $QM=PM$. Hence PD is parallel to CV, whence, and from the bisection of QD, PD is bisected in M_2: it is therefore an ordinate to CB.

(43.) Prop. If two parallel tangents to an ellipse or hyperbola meet any two other parallel tangents, the *segments* between *contact* and *concourse* are respectively equal to the *corresponding segments* of the other tangents *parallel to them.*

Let the parallel tangents VT, UT_2 meet the other parallel tangents TPT_2, KT_3 then $TP=KT_3$; and $VT=UT_3$ &c.

For joining CT, if it be produced to meet VT_3 we have the sides $CV=CU$, and the $<$s at C and at V and U equal. Whence $CT=CT_3$ and $VT=UT_3$.

In like manner, if CT_3 be produced to meet TP we have $CT = CT_3$ and $KT = T_3P$; it therefore meets both PT and VT in T their point of concourse.

(44.) PROP. The same construction remaining, and drawing a diameter parallel to one pair of tangents, the semi-diameter)^2=*rectangle* of the *parallel tangents*, or of the *segments* of one of them.

Draw the diameter CB parallel to the tangents VT UT_2. Let TP meet the diameter UV produced in τ, and CB produced in τ_2; draw the ordinates PM, PM_2 to the two diameters. Then we have

$$V\tau U = \left\{\begin{matrix} \text{(ell.)}\ C\tau^2 - CV^2 \\ \text{(hyp.)}\ CV^2 - C\tau^2 \end{matrix}\right\} = (31.) \left\{\begin{matrix} C\tau^2 - C\tau . CM \\ C\tau . CM - C\tau^2 \end{matrix}\right\}$$

$$= C\tau M$$

$$\text{Whence } \frac{\tau M}{V\tau} = \frac{U\tau}{C\tau} = (\text{by } \|\text{s}) \frac{PM = CM_2}{VT} = \frac{UU_2}{C\tau_2}.$$

$$\therefore VT.\ UT_2 = C\tau_2 .\ CM_2 = CB^2 \text{ (by 31.)}$$

Which by the above also $= TUT_3$.

SECTION II.

THE CONSTRUCTION OF THE CURVES BY THE FOCUS, AND PROPERTIES REFERRING TO THIS CONSTRUCTION.

PARABOLA.

DEF. The *focus* is that point in the axis where it is cut by the *latus rectum.*

The tangent at the extremity of this ordinate is called the *focal tangent.*

A perpendicular to the axis at the point where the focal tangent meets it, is called the *directrix.*

A line drawn from any point in the curve to the focus is called the *focal line.*

The abscissa cut off on the axis by the latus rectum is called *the focal abscissa.*

PROP. The focal abscissa= $\frac{1}{4}$ latus rectum.

(Fig. 12.) F being the focus, we have (by sect. i. 4.)

$$LF^2=L.\ VF\ \therefore=2LF.\ VF$$

$$\therefore\ VF=\tfrac{1}{2}LF=\tfrac{1}{4}L.$$

(2.) PROP. From any point in the curve, a perpendicular being drawn to the directrix, and a line to the focus,

The perpendicular = the focal line.

1st. The tangent at L meeting the vertical tangent in T from (i. 24), by parallels we have

$$VT=VF.$$

2dly. The directrix being constructed according to the definition, and through any point P drawing the ordinate produced to meet the tangent, tPM, and joining PF, we have (by i. 13.)

$$\frac{TL^2}{Lt^2} = \frac{VT^2 \; (\;=)\; VF^2}{PtQ \; (\therefore = \; FM^2}$$

And adding equals .... $+PM^2$ ... $+PM^2$

$$\overline{=tM^2} \; (\therefore =) \; \overline{FP^2}.$$

Hence (denoting a perpendicular on the directrix from any point, as P, &c. by ΔP, &c.) we have

$$FP = \Delta P \qquad LF = \Delta L \qquad VF = \Delta V.$$

Cor. 1. If at any point (as suppose t) we have $tF > t\Delta$, *the point* t *lies out of the curve and above it.*

For conceive a circular arc with centre F and distance Ft to cut the curve in q, then we have

$$tF = qF = q\Delta \; \therefore > t\Delta.$$

and consequently q lies below t, or t is without the curve.

(3.) Cor. 2. The *tangent* at any point in the curve makes *equal angles* with the *focal line* and the *diameter* through that point.

(Fig. 15.) PF being any focal line, suppose PH, the diameter produced and meeting the directrix in H. Draw τPt bisecting the angle FPH; in it take any point t, join tF, tH; then from the common side tP, the equal angles at P, and the equal sides PF PH, the third sides of the triangle are equal; or $tF = tH > t\Delta$. $\therefore$ t is a point out of the curve, and above it; or $tP\tau$ is a tangent. And the $< FPH$ being bisected, it is evident that $< FP\tau = tPM$.

(4.) Cor. 3. From the bisection of the angle by the tangent we have, (by parallels,) if the tangent meet the axis in T, $< FPT = FTP \therefore FP = FT$.

Also, $F\tau$ being the focal perpendicular, TP is bisected in τ; and since, from (i. 24.) and parallels, TP is also bisected by its concourse with the vertical tangent, $\therefore \tau$ is the point of concourse.

Def. The *Normal* is a perpendicular to a tangent at the point of contact, terminated by the axis.

The *Subnormal* is the segment of the axis intercepted between the normal and an ordinate through the point of contact.

The *focal perpendicular* is the perpendicular upon a tangent from the focus.

(5.) Prop. The normal = twice the focal perpendicular. By parallels and (i. 24.)

$$\frac{2\,T\tau\ (\ =)\ TP}{2\,F\tau\ (\therefore=)\ PN} \quad \frac{2\,T\tau}{2\,F\tau} = \frac{TP}{PN}.$$

(6.) Cor. 1. The subnormal = half the latus rectum.

From the Prop. and the similar triangles PMN $VF\tau$,

$$\frac{PN\ (\ =)\ 2F\tau}{NM\ (\therefore=)\ 2VF = \tfrac{1}{2}L.} \quad \frac{PN}{NM} = \frac{2F\tau}{2VF} = \tfrac{1}{2}L.$$

(7.) Cor. 2. The segment of the focal line cut off by a perpendicular from the extremity of the normal = half the latus rectum.

Drawing the perpendicular NK, the side PN being common, and $\angle FPN = FNP$ by (4), from the equal triangles we have

$$PK = NM = \tfrac{1}{2}L.$$

(8.) Prop. $\dfrac{\text{Focal line}}{\text{Focal perpendicular}} = \dfrac{\text{Focal perpendicular.}}{\text{Focal abscissa.}}$

From the right angles at V and τ we have similar triangles, whence,

$$\frac{VF}{F\tau} = \frac{F\tau}{FT = FP}.$$

(9.) Hence $F\tau^2 = VF \, . \, FP \qquad F\tau^2 \propto FP$

Or the *square of the focal perpendicular varies as the focal line.*

(10.) Prop. The distance from the *vertex* of any diameter to the *focus* = $\frac{1}{4}$ of the *parameter* to that diameter.

By the right angled triangle we have (Euc. 6. 8.)

$NTM = UT^2 = VT \, . \, P$ (by i. 23.) and parallels.

Hence by proportionals

$$\frac{TM}{TV = \frac{1}{2}\,TM} = \frac{P}{NT = 2UF \therefore \frac{1}{2}P.}$$

(11.) Cor. Let PM be the ordinate to the same diameter which passes through the focus;—then

$$PM^2 = 4\,UF \, . \, UM = 4\,UF^2 \text{ (by 4.)}$$

Or the *focal ordinate* to any diameter = its *parameter*.

THE CONSTRUCTION OF THE CURVES BY THE FOCUS; AND PROPERTIES REFERRING TO THIS CONSTRUCTION.

ELLIPSE AND HYPERBOLA.

(12.) DEF. The *focus* is that point in the axis, where it is cut by the latus rectum. (See Fig. 13, 14.)

COR. Hence there are *two* such points in the axis of the ellipse, one on each side of the centre at equal distances: and *one* in each opposite hyperbola.

The *focal tangent*, *focal line*, and *directrix* are defined as in the parabola.

COR. There is a *directrix* belonging to each vertex of the ellipse and opposite hyperbolas.

DEF. The abscissæ into which the axis is divided by the focus are called the *focal abscissæ.*

(13.) PROP. The rectangle of the focal abscissæ = (semi-conjugate axis)².

For the ordinate LF being by supposition $= \frac{1}{2}L$ by substituting it in the proportion (i. 7.) we have

$$\frac{B^2}{A^2} = \frac{LF^2}{VFU} \begin{matrix}(=) \\ (\therefore =)\end{matrix} \frac{\frac{1}{2}L^2}{B^2}.$$

(14.) PROP. From any point in the curve a perpendicular being drawn to the directrix, and a line to the focus,

$$\frac{\text{Perpendicular to directrix}}{\text{Focal line}} = \text{a constant ratio.}$$

DEF. This ratio is called the *determining ratio.*

1st. (Fig. 13, 14.) The tangent to L meeting the vertical tangents in T, T_2 (as in fig. 10.) we have,

$$\frac{VT}{UT_2} = \frac{TL}{T_2L} = (\text{by } \|\text{s}) \frac{VF}{UF};$$

Hence $VT.\ UT_2$ is similar to VFU, and each of them being $= B^2$ we have,

$$VT = VF, \text{ and } UT_2 = UF.$$

2dly. The directrix being constructed according to the definition, and through any point P drawing the ordinate produced to meet the tangent tPM, and joining PF, we have (as in the parabola)

$$\frac{TL^2}{Lt^2} = \frac{VT^2}{PtQ^2} \overset{(=)}{\underset{(\therefore=)}{=}} \frac{VF^2}{FM^2}$$

And adding equals .... $+PM^2 \quad +PM^2$

$$tM^2 \ (\therefore=)\ FP^2$$

Hence (denoting a perpendicular on the directrix from any point, as P, &c. by ΔP, &c.) we have by similar triangles,

$$\frac{\Delta V}{VT = VF} = \frac{\Delta L}{LF} = \frac{\Delta t = \Delta P}{tM = PF} = \frac{\Delta U}{UT_2 = UF}$$

This ratio is evidently,

In the ellipse $= \frac{>}{<}$

In the hyperbola $= \frac{<}{>}$

(15.) COR. 1. The *sum,* in the ellipse, and the *dif-*

ference in the hyperbola of the *distances* of any point from the *two foci,* is equal to the *axis.*

For taking the directrix to the second vertex, and adding or subtracting the antecedents and consequents of the ratios = the determining ratios, we have,

$$\frac{\Delta V \pm \Delta U}{VF \pm UF}\ \begin{matrix}(=)\\=\\(\therefore =)\end{matrix}\ \frac{\Delta P \pm P\Delta_2}{FP \pm FP_2}$$

(16.) Cor. 2. Hence in the ellipse $FB = A$ (Fig. 16. 17.)

$$\text{and } CF_2 = A^2 - B^2$$

Or the line joining the *focus* and *conjugate vertex* is equal to the *semi-axis.*

In the hyperbola, (joining the conjugate vertices)

$$B^2 = VFU = CF^2 - CV^2$$
$$\text{or } CF^2 = A^2 + B^2$$
$$\therefore CF = VB \text{ (Euc. 47. 1.)}$$

Or the line joining the *conjugate vertices* is equal to the distance of the *focus* from *the centre.*

(17.) Cor. 3. The *conjugate hyperbolas* having the same axes, and VB being the same in reference to each, and the above property applying also, their *foci* are at the *same distances* from the *centre* as the foci of the first pair of curves.

(18.) Cor. 4. In the equilateral hyperbola $VFU = A^2$.

(19.) Cor. 5. If at any point (as suppose t) we have in the ellipse the *sum* of its focal lines *greater,* or in the hyperbola their *difference less* than the axis, the point t lies *out of the curve and above it.*

For conceive a circular arc with centre F and distance Ft to cut the curve in q, then we have

Ell. } Hyp. } $F_2t \pm Ft \gtrless 2A$

Also $F_2q \pm (Fq =) Ft = 2A$

$\because F_2q \lessgtr F_2t$ or q lies below t, and t is above the curve.

(20.) Cor. 6. The *tangent* at any point in the curve makes *equal angles* with the *focal lines* from that point.

In PF_2, or PF_2 produced, take $PH = PF$, draw a line through P, making equal angles with the focal lines; in it take any point t and join tF, tF_2, tH; then from triangles, $tH = tF$

Whence (in the ellipse directly,)

(And in the hyperbola, in tF_2 taking $tk = tH$ and observing that since tkH is an isosceles triangle, the $\angle\ tkH$ is acute, and $\therefore\ \angle\ FkH$ obtuse, we have $F_2H > F_2k$:—)

Ellipse, $tF_2 + tH > PF_2 + PH = 2A$

Hyperbola, $tF_2 - tH < PF_2 - PH = 2A$

Therefore (19) any such point t lies out of and above the curve, or Pt is the tangent at P.

(21.) Prop. Drawing a tangent at any point in the curve, and perpendiculars upon it from the foci,

The distance from the *centre* to the *right angle* is equal to the *semi-axis*.

(Fig. 16, 17, 18, 19.) Designating by τ the point of concourse of the tangent with the focal perpendicular; join $C\tau$, and producing F_2P, take $PH = PF$. Then from the bisection of the angle by the tangent we have

$$tH = tF \text{ also } CF = CF_2$$
$$\therefore\ C\tau \text{ is parallel to } F_2H$$
$$\therefore\ C\tau = \tfrac{1}{2}\, F_2H = A.$$

(22.) Cor. 1. Drawing the diameter parallel to the tangent, meeting the focal line in E, we have also by parallels

$$C\tau = PE.$$

Or the *semi-axis* is equal to the *segment of the focal line* cut off by the *semidiameter parallel* to the tangent.

(23.) Cor. 2. The proposition holds good with either focal perpendicular. Hence $\tau\tau$ are points in a circle on VU: and producing $F_2\tau$ till it meets $C\tau$ produced in S, $F_2S = \tau H = \tau F$:

Or a *circle* described on the *axis* passes through the *extremities of the focal perpendiculars.*

(24.) Cor. 3. Since $C\tau = CS$, S is a point in the circle, whence $F_2\tau\,.\,(F_2S =)\ F\tau = VFU = B^2$.

Or *the semiconj. axis*$)^2$ = *the rectangle of the focal perpendiculars.*

Def. The *normal* and *subnormal* are defined, as in the parabola.

Def. The *central perpendicular* is the perpendicular drawn from the centre upon a tangent.

Cor. The central perpendicular is equal to the segment of the normal cut off by a diameter parallel to the tangent, or $PG = CR$.

(25.) Prop. The normal $\times$ the central perpendicular = semiconj. axis$)^2$.

For from parallels and similar triangles we have

$$\frac{F_2\tau}{CR = PG} = \frac{F_2P}{C\tau} = \frac{PN}{F\tau}.$$

$$\therefore \text{(by 24) } PG \,.\, PN = B^2$$

Or the normal $= \frac{B^2}{PG}$, or $= \frac{F\tau \,.\, F_2\tau}{PG.}$

(26.) Prop. Drawing an ordinate to the axis from any point

$$\frac{\text{The subnormal}}{\text{Dist. from centre to ordinate}} = \frac{\frac{1}{2}\text{ latus rectum}}{\text{semi-axis.}}$$

For by the right angled triangle PNM (Euc. VI. 8.) and (i. 33.)

$$\frac{NM}{CM} = \frac{NM \,.\, \text{subtan.} = PM_2}{CM \,.\, \text{subtan.} = VMU} = \frac{\frac{1}{2}L}{A}$$

Or the subnormal $= \frac{1}{2} L \,.\, \frac{CM}{A}$

(27.) Cor. The segment of the focal line by a perpendicular from the extremity of the normal is equal to half the latus rectum.

(Fig. 16. 18.) For drawing NK perpendicular to FP, we have by similar triangles,

$$(PE =)\ A \,.\, PK = PN \,.\, PG = B^2 = A \,.\, \tfrac{1}{2} L$$

$$\therefore PK = \tfrac{1}{2} L.$$

(28.) Prop. Drawing any tangent, (as in the last Prop.) and drawing also the vertical tangents to the axis, meeting it;

A *circle* described on *the tangent* thus limited, *passes through the foci.*

(Fig. 17. 19.) Joining VT, VT_2, we have, since $VT \cdot UT_2 = VFU$

$$\therefore \frac{VT}{VF} = \frac{UF}{UT_2}$$

Therefore the triangles VTF UT_2F are similar: and being right angled, we deduce the $< TFT_2$ a right angle; and therefore in a semicircle.

In the same way with the other focus.

(29.) Cor. 1. Hence drawing a diameter parallel to the tangent

$$FPF_2 = TPT_2 = D^2 \text{ (by i. 44.)}$$

Or the *parallel semidiameter*)^2 = *rectangle of focal lines.*

(30.) Cor. 2. $\dfrac{\text{Focal line}}{\text{Focal perpendicular}} = \dfrac{\text{Parallel diameter}}{\text{Conj. axis.}}$

For by similar triangles we have,

$$\frac{F\tau}{FP} = \frac{F_2\tau}{F_2P} \therefore \frac{F\tau^2}{FP^2} = \frac{F\tau \,.\, F_2\tau}{FP \,.\, F_2P} = \frac{B_2}{D^2}$$

$$\text{Hence} \quad F\tau^2 = B^2 \,.\, \frac{FP}{F_2P}$$

In the ellipse FP increases while F_2P diminishes

$$\therefore F\tau^2 \propto > FP$$

In the hyperbola FP increases while F_2P increases

$$\therefore F\tau^2 \propto < FP$$

Or the square of the focal perpendicular varies in the ellipse *more*, and in the hyperbola *less* than the focal line.

(31.) Prop. The semi-axis × semi-conj. axis is equal to any semidiam. × the central perpendicular upon the tangent parallel to it.

For since $C\tau$ is parallel to HF_2 from the bisection of the angle FPH we have, by (30.)

$$\frac{B}{D}=\frac{F\tau}{FP}=(\text{sim.}\,\triangle)\frac{CR=PG}{C\tau=A}$$

$$\therefore A.B=PG.D.$$

(32.) Cor. 1. In the *equilateral hyperbola* this becomes

$$B^2=PG.D=\text{ (by 25) } PG.PN$$
$$\therefore PN=D.$$

Or the *normal* to any point=*the semidiameter.*

(33.) Cor. 2. *The areas of the parallelograms circumscribed about any conjugate diameters are equal.*

Since any such area$=4.(PG.D)=4.A.B.$

(34.) Prop. The *sum* in the ellipse, and the *difference* in the hyperbola, of the *squares of the axes*, is equal to the *sum or difference* of the *squares of any conjugate diameters.*

By Euc. II. 12. Since the triangles FPC F_2PC have equal bases and a common altitude, and since in the two curves (16) we have $CF^2=A^2\mp B^2$, and (writing $CP=G$),

$$FP^2+F_2P^2=2G^2+(2CF^2=)\ 2A^2\mp 2B^2$$

Adding or subtracting equals

$$\pm 2FPF_2=\pm 2D^2 \text{ (by 29.)}$$

$$\overline{(FP\pm F_2P)^2=4A^2=2G^2\pm 2D^2+2A^2\mp 2B^2}$$

$$\therefore 2A^2\pm 2B^2=2G^2\pm 2D^2$$

(35.) Cor. In the *equilateral* hyperbola, since $A=B$,

$$2G^2-2D^2=0 \therefore G=D.$$

Or *any conjugate diameters are equal.*

(36.) Cor. 2. In the *equilateral hyperbola* D being a diameter through any point P, and CR the central perpendicular upon the tangent at the same point, the conjugate diameters being equal, we have from (31)

$$CR \,.\, D=A^2.$$

(37.) Cor. 3. In the *equilateral hyperbola*, drawing an ordinate PM to the axis through P, and the tangent meeting the axis at T, (see Fig. 19.) we have,

$$\frac{CM}{CP}=\frac{CM \,.\, CR}{CP \,.\, CR=A^2=CM \,.\, CT}=\frac{CR}{CT}.$$

Or the triangle CPM is similar to CRT, and $<RCM=RCT$. And since the triangle CPM is similar, and equal to that formed in like manner by the diameter through the opposite extremity of the ordinate produced, it follows that the *central perpendicular coincides with the diameter* through the extremity of an ordinate to the axis, opposite to that at which the tangent is drawn.

Or, in other words, if through the opposite *extremities of any ordinate to the axis* be drawn *a diameter* and *a tangent, they are at right angles.*

(38.) Prop. Drawing an ordinate to any diameter *through the focus*, this *focal ordinate* is a *third proportional* to the *conjugate diameter* and the *axis.*

(Fig. 17. 19.) Taking a focal ordinate to the diameter CP or D, and calling the conjugate G, by (i. 34, 36.) we have,

$$\frac{PM^2}{\left.\begin{array}{l}\text{Ell. } D^2 - CM^2\\ \text{Hyp.} CM^2 - D^2\end{array}\right\}} = \frac{G^2}{D^2}$$

But by parallels and proportionals

$$\frac{\left.\begin{array}{l}\text{Ell. } D^2 - CM^2\\ \text{Hyp. } CM^2 - D^2\end{array}\right\}}{\left.\begin{array}{l}\text{Ell. } PE^2 - EF^2\\ \text{Hyp. } EF^2 - PE^2\end{array}\right\} = F_2PF = G^2 \text{ (by 29.)}} = \frac{D^2.}{PE^2 = A^2}$$

(by 22.)

Therefore *ex æquo*, and extracting the roots

$$\frac{PM}{G} = \frac{G}{A}.$$

With the axis the focal ordinate and parameter coincide; in other diameters they are different: some writers call the focal ordinate the *parameter*.

(39.) Def. The distance from the centre to the focus is called the *excentricity*.

Prop. The ratio $\frac{\text{semi-axis}}{\text{excentricity}}$ = the determining ratio.

For the construction being as in (14.) (Fig. 13, 14.)

$$\frac{P\Delta}{PF} = \frac{\frac{1}{2}(\Delta_2V \mp (\Delta U =)\Delta_2U =)VU = CV}{\frac{1}{2}(F_2V \mp VF =)\ F_2F \qquad = CF}$$

(40.) Cor. Hence by proportionals,

$$\frac{P\Delta^2}{\left\{\begin{array}{l}P\Delta^2 - PF^2 \text{ (ell.)}\\ PF^2 - P\Delta^2 \text{ (hyp.)}\end{array}\right\}} = \frac{CV^2}{\left\{\begin{array}{l}CV^2 - CF^2 \text{ (ell.)}\\ CF^2 - CV^2 \text{ (hyp.)}\end{array}\right\}} = \frac{A^2}{B^2}$$

Whence by taking the same ratios in another curve, it appears that if two ellipses or two hyperbolas have the *same determining ratio,* their *axes* are also in the *same ratio.*

(41.) Prop. From the construction of the hyperbola by the directrix we obtain an easy mode of solving the celebrated problem, *to trisect a given circular arc geometrically.*

(Fig. 36.) Let FG be the arc to be trisected: drawing the diameter $\Delta\Delta \perp$ chord, trisect ΔF; then with F as a focus, $\Delta\Delta$ a directrix, and determining ratio $= \frac{2}{1}$, describe an hyperbola cutting the arc in P:

$$\frac{VF}{F\Delta} = \frac{2}{1} = \frac{FP}{P\Delta = \frac{1}{2}PQ} \therefore FP = PQ = QG,$$

or P, Q are the points of trisection.

PART II.

COMPRISING THE PROPERTIES OF ASYMPTOTES—CURVATURE—AREAS, &c.

SECT. I.

PROPERTIES REFERRING TO THE ASYMPTOTES OF THE HYPERBOLA.

DEFINITION. The diagonals of the circumscribed rectangle produced indefinitely are called *asymptotes.*

(1.) PROP. The asymptotes never meet the curve, though it approaches continually nearer to them.

(Fig. 20.) From (Part I. sect. i. art. 35.) $VL^2 = PRQ$. Hence at *any point R* there must be *an interval PR,* and if *R* be infinitely remote from the centre, *RQ* is infinitely great, and therefore *PR* infinitely small.

(2.) COR. The same lines are asymptotes to the conjugate hyperbolas, since the rectangle about *the axes,* and consequently its diagonals, is the same for the conjugate hyperbolas.

(3.) Scholium. The asymptotes may also be defined as constructed by a reference to the cone, in the following manner:

Let there be a vertical section of the cone parallel to the plane of the hyperbola. Let planes touch the cone along the sides of this section; the intersections of these planes with the plane of the hyperbola are the *asymptotes* to the hyperbola, as is easily shewn by means of the circles parallel to the base.

(4.) Cor. 1. The angle which the asymptotes form with each other (towards the first pair of curves) will evidently be,

$$\left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} \text{a right angle as } A \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} B.$$

Hence equilateral hyperbolas are sometimes called rectangular.

(5.) Cor. 2. Since from (I. i: 29.) the tangent at all finite distances meets the axis between the vertex and centre, the centre is the limiting position of its concourse with the axis; but its corresponding limit of the point of contact is at an infinitely remote part of the curve. Hence *the tangent* has for its *limiting position* that of *coincidence* with the *asymptote*.

(6.) If any ordinate be produced to meet the asymptotes, the segments between the curve and asymptote on each side are equal. It appears in the demonstration of (I. i. 35.) that $R_2P = Q_2\rho_2$.

(7.) From any point in the curve let lines be drawn to the two asymptotes, and from any other point lines to the asymptotes parallel to the first: then the rectangle of the lines from the first point = that of the parallels from the second.

(Fig. 21.) Taking any points P, Q, draw $PS \parallel QS$ and $P\Sigma \parallel Q\Sigma$; then by sim. △s

$$\frac{QS}{PS} = \frac{QR = P\rho}{PR = \rho Q} = \frac{P\Sigma}{Q\Sigma} \therefore QS \,.\, Q\Sigma = PS \,.\, P\Sigma.$$

(8.) If these lines are respectively parallel to the asymptotes, the parallelograms formed are equal; since the sides about the = angles are reciprocally proportional.

(9.) Def. *Asymptotic segment, or abscissa.* The segment between the centre and any point in the asymptote.

Asymptotic secant, or ordinate. A line from any point in one asymptote to the curve, and parallel to the other.

(10.) The asymptotic segments are inversely as the asymptotic secants.

$$\text{For by (7)} \frac{PS_4}{QS_2} = \frac{Q\Sigma = (\text{by}\parallel\text{s})\ CS_2}{P\Sigma = (\text{by}\parallel\text{s})\ CS_4}.$$

(11.) Cor. 1. This property enables us to prove a point before alluded to. (I. i. 41.)

From (I. i. 38.) The lines joining the vertices of any conjugate diameters are parallel to one asymptote and bisected by the other; consequently their *halves* on one side of the asymptote *are asymptotic secants to the first curve*, and (by the above) are therefore inversely as the segments:

But their equals on the other side are in the same ratio:

Also the asymptotic secant through the vertex of the *axis* has its other extremity *in the conjugate curve:*

Hence the extremities of all the parallels to it, that is, the *extremities of the conjugate diameters*, lie in the *conjugate curve.*

(12.) Cor. 2. The asymptote is the limit between those diameters which meet the first pair of curves, and those which meet the conjugate curves.

(13.) Cor. 3. No diameter can be parallel to an asymptote. And since every tangent is parallel to the diameter conjugate to that through the point of contact, *no tangent can be parallel to an asymptote.*

(14.) DEF. *Hyperbolic sector.* The figure contained by two semi-diameters and the intercepted curve.

Hyperbolic trapezium. The figure contained between two asymptotic secants and the intercepted curve.

(15.) The sector = the trapezium on the same curve.

For $\triangle\ CS_2Q = SCQ_2$, each being $= \frac{1}{2}$ the equal parallelograms (8). $\therefore$ Each + curvilinear area xQQ_2 − common $\Delta\ CSx$ are equal; or sector = trapezium.

(16.) Let a vertical tangent and any ordinate to a diameter be drawn; and asymptotic secants from the vertex and each extremity of the ordinate: then,

The segments are in continued proportion.

1st. D being the vertex, and PQ the extremities of an ordinate, we have

$$\frac{CS_2 = (\|\text{s})S_4R}{CS_3 = (\|\text{s})S_3T}\left(=\text{sim.}\ \triangle\,\text{s.}\right)\frac{S_4P}{S_3D} = (10)\,\frac{CS_3}{CS_4}$$

and conversely if the segments are taken, proportionals P, Q, are the extremities of an ordinate $\|$ tan. at D.

2dly. With any other $\|$ ordinate P_2Q_2 we have in the same way,

$$\frac{CS}{CS_3} = \frac{CS_3}{CS_5}\ \text{whence (by proportionals)}\ \frac{CS}{CS_2} = \frac{CS_4}{CS_5}$$

but if Q_2 be so taken that DQ_2 is $\|$ tan. at Q, we have also

$$\frac{CS}{CS_2}=\frac{CS_2}{CS_3} \text{ which, above, } =\frac{CS_3}{CS_4}=\frac{CS_4}{CS_5}.$$

(17.) The trapezia on the curves intercepted at the opposite ends of two ordinates, are equal.

$\triangle\, CMP = CMQ$, and curvilinear area $P_2PMM_2 = MM_2QQ_2$.

∴ Sector $CPP_2 =$ sect. CQQ_2, and ∴ (by 15) trapez. $PP_2 S_4S_5 = QQ_2SS_2$.

(18.) *Segments* in *geometrical* progression give *trapezia* in *arithmetical.*

Taking the segments CS, CS_2, CS_3, &c. continued proportionals, since with any segments proportionals we have $PQ \parallel DT \parallel P_2Q_2$.

∴ Trapez. $SS_2 = S_4S_5$ and $S_2S_3 = S_3S_4$; also $SS_2 = S_2S_3$.

∴ the differences are $=$, and the trapezia SS_2, SS_3, SS_4, &c. are in arithmetical progression.

(19.) Prop. A line parallel to an asymptote which meets the curve of the hyperbola, cuts it in one point, and never meets it again.

For since the curve approaches indefinitely near to the asymptote, no parallel to the asymptote can lie between it and the curve without meeting it; neither can it touch the curve (13.); consequently such a parallel must cut the curve, and being produced it never approaches the asymptote, and therefore never meets the curve, which is continually approaching nearer to the asymptote.

(20.) These lines are thus analogous to the diameters of the parabola, and they also possess a further analogous property referring to the directrix: viz.

If a line parallel to the asymptote of an hyperbola be drawn through any point to meet the directrix, its *segment between the curve and directrix is equal to the focal line* of that point.

(Fig. 35.) *CT* being the asymptote, draw the other diagonal *UB* and construct *PF*, *P*Δ, as in Part I. sect. ii.; then from (I. ii. 40) we have

$$\frac{PF}{P\Delta}=\frac{CF_{,}=UB=CT}{CU}.$$

At *P* drawing *PK* to meet the directrix, parallel to the asymptote *TC*, we have similar △s *P*Δ*K*, *CTU*; whence the ratio above becomes

$$=\frac{PK\ (\because)\ PF}{P\Delta}.$$

(21.) Since this holds good for any point in the curve, (designating by *PK*, *LK*, &c. lines from points in the curve to the directrix, *parallel* to *CT*,) at *L*, where *FL*, *LK*, lie in one line, we have $FL=LK$.

And *FQ* being the focal ordinate,

$$FQ=QK=2\,FL, \text{ or } FL=\tfrac{1}{4}L.$$

Properties exactly similar to those of the parabola.

SECTION II.

I. SIMILAR CONIC SECTIONS.

DEF. *Similar conic sections.* Two sections, such that any rectilineal figure being inscribed in one, a similar figure may be inscribed in the other.

(1) Two conic sections are *similar* if their *determining ratios* are *equal.*

(Fig. 22.) In two conic sections of the same kind, the determining ratios being equal, and P any point in one, take π in the other, so that $\angle PFM = \pi F\mu$: whence we have

$$\frac{PF}{FM} = \frac{\pi F}{F\mu} \quad \frac{PF}{\Delta M} = \frac{\pi F}{\delta\mu}$$

$$\therefore \frac{PF}{(\Delta M \mp MF =) F\Delta} = \frac{\pi F}{(\delta\mu \mp \mu F =) F\delta}$$

Again; take any other angle P_2FM, and $\pi_2F\mu$ equal to it; then, as above, we have

$$\frac{F\Delta}{F\delta} = \frac{FP}{F\pi} = \frac{FP_2}{F\pi_2} = \frac{FP_3}{F\pi_3}; \text{ \&c.}$$

$$\text{or } \frac{FP}{FP_2} = \frac{F\pi}{F\pi_2}, \quad \frac{FP_2}{PF_3} = \frac{F\pi_2}{F\pi_3}, \text{ \&c.}$$

Thus joining PP_2, $\pi\pi_2$, P_2P_3, $\pi_2\pi_3$, we have a series of $\triangle$s with one angle equal, and the sides about the equal angles proportionals. Hence the $\triangle$s are respectively similar; and therefore the whole figures made up of any number of them. Hence the two curves are by Def. similar.

(2) Cor. All parabolas are similar.

Since the determining ratio in all parabolas = 1.

(3) Cor. 2. Ellipses or hyperbolæ are *similar* if the *ratios* of their *axes are equal.*

For from (I. ii. 40.) it appears that equal determining ratios give equal ratios of axes. The curves are ∴ similar on the former principle.

(II.) VARIATION OF RADIUS VECTOR.

(1) Calling the radius vector (or focal line) R, and the angle it forms with the axis ψ,

$$\text{In the parabola, } R = \frac{\frac{1}{2}L}{1 + \cos.\ \psi}$$

(Fig. 12.) The construction being as in Part I. Sect. II. we have

$$PF = \Delta M = 2VF + (FM =)\ PF\ .\cos. \angle PFM$$
$$= 2VF - PF \cos. \angle VFP$$

$$\therefore \text{transposing and dividing, } PF = \frac{\frac{1}{2}L}{1 + \cos.\ \psi}$$

(2) In the ellipse and hyperbola, calling the semi-axis A, and the excentricity or distance from centre to focus E,

$$R = \frac{\frac{1}{2}L}{1 + \frac{E}{A}\cos.\ \psi}$$

(Fig. 13, 14.) The construction being as in (I. ii.) we have,

$$\frac{PF}{\Delta P = \Delta F + FM} = \frac{FC}{CV}$$

(By I. 32.) $CV^2 = CF^2 \pm CF \, . \, F\Delta . \therefore CF \, . \, F\Delta = B^2$

Hence, multiplying extremes and means,

$PF \, . \, CV = (\Delta F \, . \, FC =) B^2 + FC \, . \, (FM =) - PF \, .$
cos. $< VFP$

Whence transposing and dividing,

$$\therefore PF = \frac{B^2}{CV + FC \, . \, \cos . \, \psi}$$

Which dividing every term by $CV = \dfrac{\frac{1}{2}L}{1 + \frac{E}{A} \, . \, \cos . \, \psi}$

(3) Cor. 1. This last expression evidently includes that for the parabola if we substitute for $\frac{E}{A}$ its value $\frac{\infty}{\infty} = 1$.

(4) Cor. 2. In any of the curves, if we conceive PF produced to meet the curve again in Q, the angle ψ corresponding to the position QF will (by trigonometry) have its cosine negative, and the formula will become $QF = \dfrac{\frac{1}{2}L}{1 - \frac{E}{A} \cos . \, \psi}$

Hence we have,

$$\frac{1}{PE} + \frac{1}{QF} = \frac{1 - \frac{E}{A} \cos . \, \psi + 1 + \frac{E}{A} \cos . \, \psi}{\frac{1}{2}L} = \frac{2}{\frac{1}{2}L}$$

$$= \text{(by fractions) } \frac{QF + PF}{PFQ}$$

$$\therefore (QF + PF) \, \tfrac{1}{2}L = 2 \, . \, PFQ.$$

Or the *semi latus rectum* is an *harmonic mean* between the *segments* of any *chord* through the *focus*.

(III) GENERAL PROBLEM.

Three lines converging to a given point being given in length and position, to describe a conic section through their extremities.

(Fig. 23.) Join PP_2, produce, and take $\dfrac{PL}{P_2L} = \dfrac{PF}{P_2F}$

Similarly with P_2P_3 . . . . $\dfrac{P_2N}{P_3N} = \dfrac{P_2F}{P_3F}$

Join LN, and draw a perpendicular through F. Also draw perpendiculars ΔP, ΔP_2, &c.

$$\text{Then by sim. } \Delta\text{s} \ . \ \frac{PL}{P_2L} = \frac{\Delta P}{\Delta P_2} = \frac{PF}{P_2F} \ \text{\&c.}$$

Take $\dfrac{\Delta V}{VF}$ = same ratio. If this = 1, the curve is a parabola.

If it = $\dfrac{>}{<}$ it is an ellipse: in which case take $\dfrac{\Delta U}{UF}$ = same ratio.

If = $\dfrac{<}{<}$ it is an hyperbola: in which case take $\dfrac{\Delta U}{UF}$ = same ratio, and ∴ in the opposite direction.

In each case we have to find by trigonometry from the given lengths PF, &c. and the given angles at which they meet in F,

1st. The sides PP_2, P_2P_3.

2dly. The sides ΔP, ΔP_2, &c.

And ΔV VF, &c. which give the dimensions of the curve in terms of the given quantities.

SECTION III.

AREAS OF THE CONIC SECTIONS.

PARABOLA.

(By the method of Exhaustions.)

DEF. Drawing a vertical tangent to any diameter, and any ordinate, the parallelogram formed by these lines with parallels to the diameter through the extremities of the ordinate, is called *the circumscribed parallelogram* of the portion of the parabola cut off by that ordinate.

(1) Cut off an area by an ordinate to any diameter; and draw chords from the vertex to the extremities of this ordinate. Calling the area of the triangle thus formed A,

The parabolic area $= \frac{4}{3} A = \frac{2}{3}$ *circumscribed parallelogram.*

(Fig. 24.) P_2VQ_2 is an area cut off by the ordinate P_2Q_2 to any diameter VB. Bisect P_2B in E; draw PE parallel to the diameter: join P_2V Q_2V; $\triangle$ $P_2VQ_2 = A$. Then we have by (I. i. 2.)

$$\frac{VM}{VB} = \frac{PM^2}{P_2B^2} = \frac{1}{4}$$

Supposing a tangent drawn at P, we have by (I. i. 24.)

$$\frac{VM = VT = PM_2}{MB - VM = M_2E} = \frac{1}{2} = \frac{\Delta VPM_2}{\Delta VM_2E}$$

$$\therefore \triangle VPP_2 = \frac{1}{4} VP_2B$$

and $\triangle VPP_2 + VQQ_2 = \frac{1}{4} VP_2Q_2 = \frac{1}{4} A$.

In like manner we find, supposing the chords PP_2 &c. drawn,

$$\triangle P_2P + PV + VQ + QQ_2 = \tfrac{1}{4} \cdot \tfrac{1}{4} \cdot A.$$

Whence the sum of all the triangles similarly formed

$$= A + \frac{A}{4} + \frac{A}{16} + \text{\&c.} = \text{(Wood's Alg. 224.) (lim.)} \tfrac{4}{3} A.$$

But the parabolic area is also the limit of the sum of the triangles $\therefore$ it $= \frac{4}{3} A$.

or area $= \frac{4}{3} \cdot \frac{1}{2}$ circ. $\square = \frac{2}{3}$ circ. $\square$

(2) Hence the areas cut off by ordinates to different diameters, with equal abscissæ, are equal.

(Fig. 25.) Take $UM_2 = VM$, draw P_2E at right angles, and from the parallelism of the tangent Δ. P_2M_2E is similar to $FT\tau$.

$$\text{By (I. ii.)} \frac{P_2M_2^2}{PM^2} = \frac{UM_2 \cdot FU}{VM \cdot FV} = \frac{FU^2 = FT^2}{F\tau^2} =$$

$$\text{(sim. } \Delta\text{s)} \frac{P_2M_2^2}{P_2E^2 \; (\because) \; PM^2}$$

$\therefore$ by $=$ bases and altitudes $\Delta\, UP_2M_2 = \Delta\, VPM$. Whence the areas, which are respectively $= \frac{4}{3} A$ are likewise equal.

AREAS.

PARABOLA, ELLIPSE, AND HYPERBOLA,

(*By the method of prime and ultimate ratios.*)

(3) If, as in (I. i. 14.) we take any conic section, and with the same axis and vertex describe a second curve of the same kind, and draw ordinates to the two curves

through the same point, the *areas* cut off by those ordinates are in ratio of the *square roots* of the *latera recta* of the two curves.

(Fig. 26.) Taking several ordinates at equal distances, and drawing parallels to the axis through $P\ \pi$ &c. we have

$$\frac{PM}{\pi M}=\frac{P_2M_2}{\pi_2M_2}\left(=\text{(ell. and hyp.) }\frac{B}{\beta}\right)=\frac{\square PM_2}{\square\,\pi M_2}$$

$$\therefore=\frac{\square\,PM_2+\square P_2M_3+\&\text{c.}=\text{(lim.) area }VM_3P_3}{\square\,\pi M_2+\square\,\pi_2M^3+\&\text{c.}=\text{(lim.) area}VM_3\pi_3}$$

$$\text{But }\frac{PM^2=L\,.\,VM\frac{MU}{2A}}{\pi M^2=\lambda\,.\,VM\frac{MU}{2A}}$$

$$\therefore\frac{PM}{\pi M}=\frac{\sqrt{L}}{\sqrt{\lambda}}=\frac{\text{area }P}{\text{area }\pi}.$$

(4) Cor. 1. In the same ratio also are the areas or "sectors" cut off by lines from the *focus* of one of the curves to the extremities of the ordinates.

From the above we have

$$\frac{PM}{\pi M}=\frac{\Delta FPM}{\Delta F\pi M}=\frac{\text{area }VPM\pm\Delta FPM=\text{sector }VFP}{\text{area }V\pi M\ \pm\Delta\,F\pi M=\text{sector }VF\pi}$$

(5) Cor. 2. In the same ratio are the sectors similarly formed by lines from the *common centre* of the two ellipses, or hyperbolæ.

$$\frac{PM}{\pi M}=\frac{\Delta\,PMC\ \pm\text{ area }VPM\ =\text{ sector }VCP}{\Delta\ \pi MC\ \pm\text{ area }V\pi M\ =\text{ sector }VC\pi}.$$

And the same will hold good with lines similarly drawn from *any point* in the axis.

(6) Cor. 3. Let the second ellipse be a circle; or the second hyperbola equilateral, then

$$\frac{\text{area ell. or hyp.}}{\text{area} \odot \text{ or eq. hyp.}} = \frac{B}{A} = (\text{ell.}) \frac{AB = R^2}{A^2} = \frac{\text{area} \odot \text{ on } R}{\text{area} \odot \text{ on } A}.$$

(7) COR. 4. Hence the area of the first ellipse = area of a circle whose radius is a mean proportional to the axes.

(8.) COR. 5. The areas of ellipses are as the rectangles of their axes.

Since area ell. = area $\odot \propto R^2 \propto A \, . \, B$.

SECTION IV.

CIRCLES OF THE SAME CURVATURE WITH THE CONIC SECTIONS.

(1) Def. *Subtense of Contact.* Take an arc from the contact of any curve with a straight line, and at its other extremity draw to the tangent a parallel to any given line through the contact: the segment of this parallel between the curve and tangent is called the *subtense.*

(2) Def. *The circle of the same curvature* to any point in a given curve, is a circle such, that if it have a common tangent with the curve at the point, the *limiting ratio of their subtenses* through the same point will be a *ratio of equality.*

(3) Prop. There can be only one circle of curvature at the same point; and

(4) In different circles the curvature is inversely as the radius.

In Fig. 28, regarding the circle only, and πt the subtense of contact being supposed *parallel to the diameter PW,* and πP joined; from the right angles, at t and in the semicircle, and from the angle in the alternate segment, we have similar triangles πtP, πWP; whence

$$\pi t = \frac{\pi P^2 = (\text{lim}) \text{ arc } \pi P^2}{PW}$$

But in the same circle, if πP be given, this ratio is constant for all points; and in different circles it is inversely as PW.

(5) In any circle taking a chord and a small arc at its extremity, at which also a tangent is drawn and limited by a subtense, (as in Def. (1). the chord has for its limit $\frac{\text{arc})^2}{\text{subtense}}$.

Let the subtense πt be now supposed *parallel to any chord PS*: suppose πP πS joined: by parallels and alternate segment Δ $P\pi t$ is similar to πPS: hence

$$PS = \frac{\pi P^2 = (\text{lim.})\ \text{arc}\ \pi P^2}{\pi t}.$$

If we now suppose the circle to be a circle of curvature to any of the curves at P, (by Def.) πP and πt in the circle = (lim.) πP and πt in the curve.

PARABOLA.

(6) Chord of the circle of curvature through focus = the parameter to the point of contact.

(Fig. 27.) Pm being the diameter of the parabola through P, PW that of the circle of curvature; PnR the chord through the focus; we have similar Δs Pnm PFT: whence, and from (I. ii. 4.)

$$Pn = Pm = \pi t.$$

Hence (by 5) $PR = \frac{\pi P^2 = (\text{lim.})\ \pi m^2 = 4FP\ .\ Pm}{Pm}$

= parameter to P.

(7) Diameter of curvature $= \frac{\text{latus rectum} \times \text{focal dist.})^3}{\text{focal perpend.})^3}$

Drawing the focal perpendicular, by similar Δs PWR PFt, and (I. ii. 8.) we have

$$PW=\frac{PR\,.\,FP\,.\,FP.VF=4VF\,.\,FP^3}{F\tau\,.\,FP\,.\,VF}=\frac{L\,.\,FP^3}{F\tau^3}$$

(8) COR. 1. $\frac{1}{2}PW=\frac{2\,FP^2}{F\tau}=\frac{2\,FP^{\frac{3}{2}}}{VF^{\frac{1}{2}}}$.

(9) COR. 2. At the vertex, the chord and diameter of curvature coincide, and each = latus rectum.

At V, FP becomes perpendicular to the tangent: $PR=PW=4\,VF=L$.

(10) The radius of curvature $=\frac{\text{normal})^3}{(\frac{1}{2}L)^2}$.

By Part I. Sect. ii. Arts. 5 and 8, we have

$$\frac{N^3=8\,F\tau^3=8VF^{\frac{3}{2}}\,.\,FP^{\frac{3}{2}}}{(\frac{1}{2}L)^2=4VF^2}=\frac{2FP^{\frac{3}{2}}}{VF^{\frac{1}{2}}}=\frac{1}{2}\,PW \text{ (by 8.)}$$

Hence the radius of curvature is as the cube of the normal.

ELLIPSE AND HYPERBOLA.

(11) Drawing a diameter through the point, and its conjugate,

Chord of curvature through centre $=\frac{2(\text{semi-conj. diam.})^2}{\text{semi-diam.}}$

The chord through the centre PS meets the curve or opposite curve in K, and CD being the conjugate diameter, we have

$$\frac{CD^2}{CP^2}=\frac{\pi m^2=(\text{lim.})\,\pi P^2}{PmK=(\text{lim.})\,\pi t\,.\,2CP}$$

$$\therefore PS=\frac{2\,CD^2}{CP} \text{ (by 5.)}$$

(12) Diameter of curvature $=\frac{2\text{ semi-conj. diam.})^2}{\text{central perpendicular.}}$

We have similar Δs PCG PSW: whence

$$PW = \frac{CP}{PG} \cdot (PS =) \frac{2\,CD^2}{CP} = \frac{2\,CD^2}{PG}$$

(13) Cor. 1. At V this $= \frac{2\,B^2}{A} = L$;

Or, at vertex, diam. of curv. = latus rectum.

(14) Cor. 2. At B it $= \frac{2\,A^2}{B}$

Or, at conj. vertex, diam. of curv. $= \frac{2 \text{ semi-axis})^2}{\text{semi-conj.}}$

(15) Chord of curv. through focus $= \frac{2\,(\text{semi-conj. diam.})^2}{\text{semi-axis.}}$

We have similar Δs PEG PRW: whence

$$PR = \frac{PG}{PE = A} \cdot (PW =) \frac{2\,CD^2}{PG} = \frac{2\,CD^2}{A}$$

(16) Cor. At B this becomes $= 2\,A$,

Or, at conj. vertex, chord through focus = axis.

(17) Radius of curvature $= \frac{\text{normal})^3}{\frac{1}{2}L^2}$

Whence the radius of curvature is as the cube of the normal.

For taking the expression (12); from (I. ii. 31.) and multiplying by PG^2, we have

$$PW = \frac{2\,CD^2}{PG} = 2A^2B^2 \left(\frac{1}{PG^3} =\right) \text{(I. ii. 25.)}\ \frac{N^3}{B^6} = \frac{N^3}{(\frac{1}{2}L)^2}$$

(18) Cor. This also (multiplying by D^3)

$$= 2\,A^2B^2\,\frac{D^3}{A^3B^3} = L \,.\, \frac{FP^3}{F\tau^3} \text{(I. ii. 30.)}$$

Or in all the Conic Sections, diameter of curvature $= \frac{\text{latus rectum} \times \text{focal dist.})^3}{\text{focal perpend.})^3}$

(19) In the equilateral hyperbola at vertex, the curvature is the same as in the circle on the axis, (from 13.)

At other points, radius of curvature is as the cube of the semidiameter through the point, (from I. ii. 32.)

APPENDIX,

CONTAINING

DEMONSTRATIONS OF THE FUNDAMENTAL PROPERTIES WITHOUT ANY DEDUCTIONS FROM THE CONE EXCEPT THOSE REFERRING TO THE AXIS.

APPENDIX.

No. I.

(1) For the sake of those students who may prefer dispensing altogether with oblique deductions from the cone, we will here give a demonstration of the fundamental proposition in the ellipse and hyperbola, that four tangents, parallel two and two, are proportionals, which was before deduced from the cone, and from which the proportionality of the squares of the ordinates to the rectangles of the abscissæ immediately follows in the ellipse, and several other important properties in both curves.

ELLIPSE AND HYPERBOLA.

(2) Assuming the construction of the curves by the directrix, if two tangents meet, the line joining their concourse with either focus forms equal angles with the focal lines to the two points of contact.

(Fig. 30.) PT P_2T meet in T. Join TF PF P_2F PF_2.

Produce FP and make $PK = PF_2$

. . . . FP_2 . . . . . $P_2K = P_2F_2$

Join TK TF_2 TK_2.

Then by (I. ii. 15.) $FK = FK_2$.

From the bisection of the angle by the tangent and the equal sides of the triangles, we have

$$TK = TF_2, \text{ which similarly} = TK_2$$
$$\therefore \Delta\, KFT = K_2FT \text{ and } \angle PFT = \angle P_2FT.$$

In the hyperbola the demonstration is exactly the same, except that PK is cut off $= PF$, and $PK_2 = P_2F$, (Fig. 31.)

The same property might be proved in the parabola; but we shall presently shew a shorter mode of deduction without reference to it.

(2) Scholium. In order to establish the next proposition, we must have recourse to a theorem which, though usually announced in a form which may be considered not strictly geometrical, is yet in reality perfectly so; viz. that "the sides of a triangle are as "the sines of the opposite angles." It would be very easy to put this theorem into strictly geometrical language by a definition of the term *sine;* and the demonstration of it depends upon the simplest principles of the third book of Euclid.

(4) The tangents meeting each other are inversely as the sines of the angles FPT FP_2T.

$$\frac{PT}{FT} = \frac{\sin. PFT}{\sin. FPT}, \quad \frac{FT}{P_2T} = \frac{\sin. FP_2T}{\sin. P_2FT = PFT}.$$

$$\therefore \frac{PT}{P_2T} = \frac{\sin. FP_2T}{\sin. FPT}.$$

(5) Let PT be produced to meet in T_2 a third tangent parallel to TP_2, and therefore at the other vertex of the diameter through P_2. The four segments of the tangents are proportionals.

Since the focal distances form equal angles with the tangent at any point, and the lines from the vertices of a diameter to the two foci are parallel, and the vertical tangents are also parallel, hence

$$\angle FP_3T_2 = \angle FP_2T.$$

Hence we have from the above,

$$\frac{PT_2}{P_3T_2} = \frac{\sin (FP_3T_2 =) \; FP_2T}{\sin FPT_2 = \sin FPT} \therefore = \frac{PT}{P_2T}.$$

(6) We have thus established the fundamental theorem before deduced from the cone, without reference to the solid, but dependent upon the description of the curves by the directrix, and the bisection of the angle by the tangent which follows immediately from that construction. The student who prefers this method will readily perceive in what order to insert the above articles.

This method is followed in Peacock's Conic Sections, second edition.

PARABOLA.

(7) It remains now to deduce the primary property of the parabola, without reference to the cone. This is done as follows: upon the assumption of the construction of the curve by the directrix and the bisection of the angle of the focal radius with the diameter by the tangent.

This construction, however, as before given, (Part I. Sect. ii.) assumes the subtangent = 2. the abscissa. The student, therefore, who adopts this mode of treating the subject must *commence* with the construction by the directrix, *assuming that as his definition of the curve.*

(8) Any diameter of a parabola bisects its ordinates.

(Fig. 32.) *VM* is any diameter of the parabola, *VT* and *PMQ* its vertical tangent and ordinate, meeting the axis produced in *T* and *O*. *VM* produced meets the directrix in A: and a parallel through Q meets it in R, and one through P in E, all at right angles to it. Join FA meeting VT in K, and PO in G. join PF, QF. Then, since $PF=PE$, and $QF=QR$ by the construction of the curve,

A circle with centre P and distance PF touches the directrix at E.
.... Q ....:.... QF..·.. .... R.

Let the first circle cut AF in H.

From the bisection of the angle AVF by the tangent we have $AK=KF$;
and VK perpendicular to AF. $\therefore PG$, which is parallel to it, is also perpendicular; and consequently $HG=GF$. Join QH: hence $QF=QH$, or the second circle also passes through H.

$\therefore\ RA^2=FAH=AE^2$.

Whence by parallels

$$QM=PM.$$

(9) If from any point in the parabola an ordinate to any diameter, and a perpendicular on that diameter produced be drawn, then the perpendicular)^2 = abscissa $\times$ 2. distance from focus to directrix.

The construction of the last art. remaining draw PL, a perpendicular from P on the diameter produced.

Then from the right angles we have similar triangles FAB, TKF, OGF. Whence

$$\frac{FA}{FB}=\frac{FT}{FK}=\text{(proportionals)}\ \frac{OT=VM}{GK}.$$

$$\therefore\ FA.GK=VM.FB;$$

but since $FA=2FK$, and $FH=2FG\ \therefore AH=2GK$

$$\therefore\ FAH=FA.\,2\,GK=VM.\,2\,FB.$$

Hence $PL^2=AE^2=$(by $\odot$) $FAH=VM.2FB$.

(10.) COR. Hence FB being a constant quantity, the squares of any such perpendiculars are as the corresponding abscissæ.

(11.) COR. 2. Take any other ordinate to the same diameter P_2M_2, and draw the perpendicular P_2S; then the $\triangle MPL$ is similar to M_2P_2S, and we have

$$\frac{PM^2}{P_2M_2^2} = \frac{PL^2}{P_2S^2} = (\text{above}) \frac{VM}{VM_2}$$

Or *the abscissæ are as the squares of the ordinates.*

(12) We have thus deduced the fundamental property of any diameter of the parabola from the construction by the directrix. This includes the properties of the axis: and the identity of the curve, with that formed by the section of the cone, is immediately established.

This method is taken from T. Newton's Conic Sections.

No. II.

The following articles comprise a brief outline of *another mode* of deducing the principal oblique properties in the ellipse and hyperbola independent of the cone. But for a complete view of the method and its application the student is referred to "A Geometrical "System of Conic Sections for the use of the Mathe- "matical Students at the Royal Liverpool Institution," published at Cambridge, 1822. from which the following outline is taken.

ELLIPSE.

(1.) (Fig. 33.) Describe a circle on the major axis. P being any point in the ellipse join CP, and in it take any point L: *through L any secant QQ_2 passes,* and meets the axis produced in K.

Ordinates to the axis through Q and Q_2 meet the circle in RR_2.

Join KR, KR_2, these lie in one line; for we have

$$\left.\begin{matrix}\frac{KM}{MQ} \\ \frac{MQ}{MR}\end{matrix}\right\} \begin{matrix} = (\text{sim.} \triangle\text{s}) \\ = (\text{by curve})\end{matrix} \left\{\begin{matrix}\frac{KM_2}{M_2Q_2}; \\ \frac{M_2Q_2}{M_2R_2}\end{matrix}\right.$$

hence, and from the right angles at MM, the $\triangle$s are equiangular, and thence KRR_2 is one line.

Through L draw an ordinate meeting KR in H;

join CH, and let it meet the circle in S. The ordinate through P meets it also in S;

$$\text{for } \frac{MH}{ML} = \frac{M_2R_2}{M_2Q_2} = (\text{by curve})\frac{mS}{mP}.$$

From this construction then we have

$$\frac{CP_2}{CS^2} = \frac{CL^2}{CH^2} = \frac{CP^2 - CL^2 = PLD}{CS^2 - CH_2 = SHS_2}$$

and this ratio $\dfrac{CP^2}{CS^2}$ depends solely on the position of P, and is independent of the position of QQ_2.

Also

$$\frac{RHR_2 = SHS_2}{QLQ_2} = \left(\begin{matrix}\text{by sim. } \triangle \text{s} \\ \text{and comp.}\end{matrix}\right)\frac{KR^2}{KQ^2}\left\{\begin{matrix}\text{A ratio which is} \\ \text{given if } QQ_2 \text{ is} \\ \text{given in position.}\end{matrix}\right.$$

$$\therefore \frac{PLD}{QLQ_2} = \frac{CP^2}{CS^2} \cdot \frac{KR^2}{KQ^2}.$$

In the same way, if we had taken any other secant, Q_3Q_4 passing through L, we should have a ratio compounded of $\dfrac{CP^2}{CS^2}$ which remains the same, and a new ratio $\dfrac{KR_3^2}{KQ_3^2}$: or,

$$\frac{QLQ_2}{Q_3LQ_4} = \frac{KQ^2}{KR^2} \cdot \frac{KR_3^2}{KQ_3^2}.$$

Hence in general, if two straight lines meet an ellipse, the rectangle contained by the distances of their intersection from the points where one of them meets the curve, is to the rectangle contained by its distances from where the other meets the curve in a ratio, which is given if the lines are given in position.

(2.) If LQ be a tangent at the point formed by the coincidence of Q and Q, supposing it to meet another secant or tangent at L, the same demonstration will hold good taking LQ^2 instead of QLQ_2, and considering the points RR_2 in the circle also to coincide.

(3.) If we take any other secant or tangent *parallel* to QQ_2 and make a similar construction, we shall have the $\triangle$s MQK, MRK, similar to the analogous new $\triangle$s.

And hence the rectangles of the segments of the parallel secants are as the rectangles of the segments of the second pair, or the squares of the tangents; and ∴ the tangents simply are in the same ratio.

HYPERBOLA.

(4.) In the hyperbola these propositions also hold good, and may be demonstrated by an extension of the principle of the demonstration in (I. i. 36.)

In Fig. (20) suppose PQ_2 were not bisected in M_2; but conceive some other point X to be where the diameter CX, to which it is an ordinate, meets it. Then (by Euc. 2.)

$$\begin{aligned} PM_2Q_2 &= PX^2 - XM_2^2 \\ &= (R_2X^2 - R_2P\rho_2) - (R_2X^2 - R_2M_2\rho_2) \\ &= R_2M_2\rho_2 - R_2P\rho_2. \end{aligned}$$

Suppose $TG\tau$ any parallel meeting the curve in G; then by triangles

$$\frac{CM_2^2}{CG^2} = \frac{R_2M_2\rho_2}{TG\tau \quad R_2P\rho_2}.$$

Hence

$$\frac{CM_2^2 - CG^2 = GM_2K}{CG^2} = \frac{R_2M_2\rho_2 - R_2P\rho_2 = PM_2Q_2}{R_2P\rho_2 = TG\tau};$$

Or,

$$\frac{TG\tau}{CG^2} = \frac{PM_2Q_2}{GM_2K} = \text{(taking any other parallel secant)}$$

$$\frac{P_3M_3Q_3}{GM_3K}$$

In the same ratio would be the rectangles of any other two parallel secants passing through M_2M_3.

The same demonstration would hold good, if the second secant were in the opposite hyperbola.

It also holds if the points M_2M_3 lie without the curve: as in Fig. (34).

(5.) It also applies, if we conceive the secants to become tangents; the squares of the tangents, and therefore also the tangents themselves, forming the terms of the proportion.

No. III.

ON THE MECHANICAL DESCRIPTION OF THE CURVES.

(14.) The mechanical descriptions of the curves will have been obvious from I. ii. 2. for the parabola; and for the ellipse, and hyperbola from I. ii. 15.

For the parabola, a ruler having a right angle, moves along a directrix or fixed ruler. At some distant point in the perpendicular part a string is fixed, equal in length to the distance of that point from the right angle. The other extremity of the string is fixed by a pin in the point assumed as the focus. While the perpendicular is the position of the axis, the string will evidently extend as far as to half the distance between the focus and the directrix, where it is doubled back. At the point where it is doubled a pencil being placed marks the vertex; and if as the perpendicular moves parallel to itself along the directrix, the string be kept stretched by the pencil at the point where it is doubled, the portion of the string between the focus and that point will always be equal to the portion of the perpendicular between that point and the directrix, or the pencil will trace out a parabola.

The ellipse is readily described by taking two fixed points as foci, and keeping a *string of given length*, fastened at each focus, stretched by a pencil whose motion constrained by the string will trace an ellipse.

For the hyperbola a ruler revolves about one focus, and at the other a string is fastened, which is also fixed at a distant point in the ruler, and of such a length that the ruler exceeds it by the major axis. Then a variable portion of the string being always kept coincident with the ruler, the *difference* between the part of the ruler devoid of string, and the length of string between the focus and the point at which the string separates from the ruler will be constant, and the string being stretched by a pencil at that point, the pencil traces out the hyperbola.

(16.) If a similar construction be made to that for the parabola, except that the *ruler* have an *oblique* instead of a right *angle*, it is evident from II. i. 20. that a curve will be traced out which will be an *hyperbola.*

The mechanical construction of the ellipse by means of the property (I. i. 15.) is perfectly obvious. Two rulers are fixed at right angles, with a grove along each: a third moveable ruler has two pegs fixed in it which intercept a portion equal to the sum or difference of the semi-axes of the ellipse proposed: these move constantly in the groves, whilst at a distance beyond one of them, equal to the semi-conjugate axis, a pencil is fixed which traces out the ellipse. This instrument is called the trammel, or elliptic compasses.

THE

PRINCIPLES

OF THE

THEORY OF CURVES:

DESIGNED AS

AN INTRODUCTION

TO THE

VARIOUS TREATISES

ON THE

FLUXIONAL OR DIFFERENTIAL

CALCULUS.

PREFACE.

THE following introduction to the general algebraic doctrine of Curves is annexed to the Conie Sections as a distinct tract. It could not have been properly united with it, or presented as a second part without a considerable alteration and extension in the plan of both.

In its present form, however, it is hoped that it may be found useful to the student; as supplying a part of the elementary course, on which a short, simple, and systematic treatise appears to be wanting. Most writers on the Fluxional Calculus *assume* the student's previous acquaintance with the theory of Curves; but the incidental slight accounts of that theory, which may be found in several works of old established reputation, are very incomplete, and fail in giving that *systematic* view of the subject, which it is the main excellence of such investigation to afford. Whilst, on the other hand, some recent treatises, as well from their extent as their abstruseness and obscurity, are ill suited to the purposes of the learner.

In the present attempt to supply this deficiency it has been the author's object to study brevity, as far as consistent with perspicuity, and the wish to explain every thing in the most familiar manner. It must be apparent that no treatise of this kind can lay claim to much originality in its materials; but the author trusts that he shall be found to have adopted some improvements in the selection, arrangement, and form of discussion.

As to the extent of the investigation, it does not pretend in any case to go further than the mere elements of each curve, so far as they can be deduced by common algebra; the object being simply to furnish the student with those primary notions of the nature of curves, possessed of which he may proceed in an unbroken and systematic course of demonstration to their various properties deduced by the application of the Differential Calculus: examples of which are so abundantly supplied in every fluxional treatise. Wherever any locus has a geometrical construction, it is given; and the student has it in his power, if he prefer it, to commence with this construction, and deduce the geometrical properties. A few of the most remarkable of these are stated, while reference is given to sources of further information respecting others.

CONTENTS.

DIVISION II.

TRANSCENDENTAL EQUATIONS.

POLAR EQUATIONS.

INTRODUCTION.

ON INDETERMINATE EQUATIONS AND THEIR LOCI.

GENERAL PRINCIPLES.

(1.) Definition. If we have an equation involving *two* quantities x, y, supposed unknown or variable, with certain others A, B, C, &c. supposed known or constant; since such an equation will not give any one determinate value of x or y, it is called an *indeterminate equation.*

An indeterminate equation will however give results of this nature; for every value assigned to the one variable, we can deduce the corresponding value of the other, involved in a certain way with known quantities: or, in other words, we obtain *the law which connects the variation of the one variable with that of the other;* the nature of such law being dependant upon the constant quantities, and the powers and combinations of the variables, in the given equation.

Indeterminate equations may be applied to various descriptions of problems; but our present object is to consider only one such application, which is comprised in the two following cases:

(2.) 1st. Let there be assumed two straight lines of indefinite length cutting each other at a given angle; we may then suppose the *two variables,* in a given indeterminate equation, to represent *portions measured off upon these given lines* from the point of their intersection; such that, assuming any portion of one line as a particular value of x, the portion of the other line shall represent the corresponding value of y as determined by the given equation.

(Fig. 1.) DEF. Two fixed lines thus assumed are called *axes*, as the lines XX, YY. The point of intersection O is termed *the origin*. The angle at which they are inclined is called *the angle of ordination*, represented by the symbol $< \omega$ and the portions xy measured off respectively on the axes, which represent corresponding values of x and y in the given equation, are termed *coordinates*.

Hence it is evident, that if, assuming any corresponding values of x and y we draw through their extremities lines parallel to the axes respectively, we shall have a parallelogram, the sides of which are equal to the values of x and y; and which being given in length and position *determine the position of the point* p in the plane of the axes. If other successive values were taken, we should have a succession of points pp_2, &c. determined in position. If a number of such points be conceived determined, indefinitely near to each other, they will lie in a certain line, straight, or curved, dependent upon the conditions of the equation determining the values of the variables: *such a line traced out by the successive positions of the point* p *is termed* THE LOCUS *of the given equation*; and an equation applied to determine in this manner the locus, is called *an equation of coordinates* [a].

(3.) 2dly. The position of any point p in a plane

[a] The application of indeterminate equations in this way to construct curves of all kinds was the invention of Des Cartes.

may be determined either by measuring its *distance* in *two* given directions, (as above,) or its *distance* along *one* given *line*, forming a known *angle*, with some other given line.

If then in an indeterminate equation one of the variables represent a *distance* measured along a line, and the other the *angle* formed by that line with another given in position, we shall equally be able to trace out the locus.

Thus (Fig. 2.) let OX be a fixed line given in position, and about the point O let the line Op revolve, forming with it a variable angle, and at the same time varying in length. The point O is termed *the pole*; the variable length *the radius vector*, designated by the symbol r; and the variable angle by the symbol $\angle \theta$. An equation expressing the relation between these two variables is called a *polar equation*[b].

In applying equations to coordinates it will be understood, according to the common principle, that if values of x measured on one side of O are considered $+$, those measured the other way are $-$. And similarly, values of y *above* O are $+$, and those below $-$.

These two kinds of equations are often expressed briefly and generally by the adoption of the term *function*, to signify an expression whose changes of value depend wholly upon the values given to one or more variables involved in it. This is commonly represented by the symbol f placed before the variable, and distinguished from a coefficient by brackets. Thus an equation of coordinates is expressed by the symbols $y=f(x)$.
A polar equation . . . . $r=f(\theta)$.

[b] The variables r and θ are by some writers called *polar coordinates*; but the use of the term appears improper, and likely to lead to confusion.

The ordinate, in the one case, being a function of the abscissa; the radius vector, in the other, a function of the variable angle.

ON THE TRANSFORMATION OF COORDINATES.

(4.) If we have a point whose position is determined by being referred by coordinates to given axes, it is evident that it may at the same time also be considered as referred by other coordinates to any other axes.

If we have given an equation expressing the coordinates of a locus as referred to one system of axes, it is often desirable to find its equation as referred to another system; we have therefore to shew how such a second equation may be deduced from the first. This operation is called the transformation of coordinates.

(5.) (Fig. 3.) Let

The first axes be X, Y; origin O; angle of ordination $\angle \omega$:

The second axes X_2, Y_2; origin O_2; angle of ordination $\angle \omega_2$:

The coordinates of p referring to the first axes; x, y:

. . . . the second axes; x_2, y_2:

The coordinates of the second origin, ... x_0, y_0.

Also let the angle formed by two lines, as x and y_2, &c. be expressed by writing $\angle xy_2$, &c.

Then the object is to find expressions for x and y in terms of x_2, y_2, and the given angles, or the converse; which may be done thus:

Drawing the parallels pqr, mn; also $O_2\, nr$, mq; we have

$$y = y_\circ + qr + qp$$
$$x = x_\circ + O_2 n + nr.$$

And the sides being as the sines of the opposite angles, we have

In the $\triangle\, O_2 mn \left\{ \frac{mn = qr}{x_2} = \frac{\sin. xx_2}{\sin. \omega}, \quad \frac{O_2 n}{x_2} = \frac{\sin. x_2 y}{\sin. \omega} \right.$

$$\triangle\, pmq \left\{ \frac{qp}{y_2} = \frac{\sin. xy_2}{\sin. \omega}, \quad \frac{nr = mq}{y_2} = \frac{\sin. yy_2}{\sin. \omega}; \right.$$

Whence substituting these values of qr, qp, $O_2 n$, nr, we deduce

$$y = y_\circ + \frac{x_2 \sin. xx_2 + y_2 \sin. xy_2}{\sin. \omega}$$

$$x = x_\circ + \frac{x_2 \sin. x_2 y + y_2 \sin. yy_2}{\sin. \omega}.$$

These formulæ correspond to the most general case; viz. where the second axes have a different origin; are inclined at a different angle, and are neither of them parallel to the first.

According to changes in these conditions the expressions will be modified; as in the following cases:

(6.) 1st. If both the second axes be parallel to the first, we have $\angle\, xx_2 = 0$, $yy_2 = 0$, $\omega = \omega_2$: whence, substituting, the formulæ become

$$y = y_\circ + y_2, \quad x = x_\circ + x_2.$$

2dly. If the origin be not removed, (retaining the other conditions in their most general form,) the only alteration will be, that we have

$$x_\circ = 0, \; y_\circ = 0.$$

We will suppose this modification to continue in the following cases.

3dly. If the *first axes be rectangular*, the denominator $\sin. \omega = 1$.

If at the same time the *second axes be oblique*, (π representing a semi-circumference,) we have

$$\angle yy_2 = \frac{\pi}{2} - xy_2, \quad x_2y = \frac{\pi}{2} - xx_2.$$

On which consideration we substitute the cosines of the complements, and obtain

$$y = x_2 \sin. x_2x + y_2 \sin. xy_2$$
$$x = x_2 \cos. x_2x + y_2 \cos. xy_2.$$

4thly. If *both* systems be *rectangular*,

$$\angle \omega = \omega_2 = \frac{\pi}{2}, \quad xy_2 = \frac{\pi}{2} - xx_2, \quad xx_2 = yy_2.$$

Whence $y = x_2 \sin. xx_2 + y_2 \cos. xx_2$
$x = x_2 \cos. xx_2 + y_2 \sin. xx_2.$

5thly. If the *first axes* be oblique, and the *second rectangular*,

$$\omega_2 = \frac{\pi}{2}, \quad xy_2 = \frac{\pi}{2} - xx_2, \quad yy_2 = \frac{\pi}{2} - x_2y$$

$$\therefore y = \frac{x_2 \sin. xx_2 + y_2 \cos. xx_2}{\sin. \omega.}$$

$$x = \frac{x_2 \sin. x_2y + y_2 \cos. x_2y}{\sin. \omega.}$$

TRANSFORMATION OF POLAR EQUATIONS.

(7.) If the nature of the locus be expressed by means of a polar equation, it is evident that it may be also considered as referred to given axes by coordinates: or if, on the other hand, its equation of coordinates be given, it may also be conceived as generated by a polar equation. It is often desirable, when the equation of coordinates is given, to obtain the polar equation or the converse; and this may be effected on the following principles.

(Fig. 2.) Conceive the point p determined by a polar equation referring to the pole O and the radius Or, to be also referred by the coordinates xy to the axes XY inclined at an angle ω: then we have,

$$\frac{x}{r} = \frac{\sin.(\omega-\theta)}{\sin.(\pi-\omega) = \sin.\omega}$$

$$\frac{y}{r} = \frac{\sin.\theta.}{\sin.(\pi-\omega) = \sin.\omega}$$

$$\text{or } x = r.\frac{\sin.(\omega-\theta)}{\sin.\omega} \quad \ldots\ldots \quad (A)$$

$$y = r.\frac{\sin.\theta}{\sin.\omega} \quad \ldots\ldots\ldots \quad (B)$$

We have thus obtained values of x and y in terms of r and θ. If then an *equation* of *coordinates* were *given*, by substituting these values it would be *transformed into a polar equation.*

We may also obtain values of r and θ in terms of x, y, and ω.

From p conceive a perpendicular drawn to the axis Y, and let the portion intercepted beyond y be z. Then by Euc. ii. 12. we have,

$$r^2 = y^2 + x^2 + 2\,y\,z$$

$$\text{but } z = x\,.\cos.\omega$$

$$\therefore r = \sqrt{(y^2 + x^2 + 2\,y\,x\cos.\omega)} \quad \ldots\ldots \quad (C)$$

Again, from the expressions before given, we deduce,

$$\frac{x}{y} = \frac{\sin.(\omega-\theta)}{\sin.\theta} = \frac{\sin.\omega\cos.\theta - \sin.\theta\cos.\omega}{\sin.\theta}$$

$$= \sin.\omega\frac{1}{\tan.\theta} - \cos.\omega.$$

$$\therefore \frac{x + y \cos. \omega}{y . \sin. \omega} = \frac{1}{\tan. \theta} \text{ or } \tan. \theta = \frac{y . \sin. \omega}{x + y \cos. \omega}$$

$$\text{or } \theta = \tan.^{-1} \left(\frac{y . \sin. \omega}{x + y \cos. \omega}\right) \quad \ldots\ldots\ldots \quad (D)^{c}$$

Thus *having* the *polar equation*, by substituting these values, we obtain an *equation of coordinates*.

If the coordinates be rectangular, $\omega = \frac{\pi}{2}$, and these expressions become,

$$x = r \cos. \theta \qquad y = r \sin. \theta$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan.^{-1} \frac{y}{x}$$

(8) To transform a *polar* equation to an equation of *coordinates*, one of whose axes forms a *given angle* with the fixed axis of the polar equation, and whose *origin* is different from the pole.

Let the second axis X_2 form an angle γ with the first fixed axis, and let the angle of ordination be ω_2. Supposing the origin to continue the same, the polar equation is readily transferred to these axes by substituting in the formulæ (A) and (B), ω_2 and $(\theta + \gamma)$; and supposing x_o y_o to be the coordinates of the second origin O_2, by Art. (6) case 1st, we have,

$$x = x_o + r \frac{\sin. [\omega_2 - (\theta + \gamma)]}{\sin \omega_2}$$

$$y = y_o + r \frac{\sin. (\theta + \gamma)}{\sin. \omega_2}$$

If the second fixed axis be parallel to the former, $\gamma = o$.

If the axes are rectangular, the expressions become

$$x = x_o + r \cos. (\theta + \gamma)$$
$$y = y_o + r \sin. (\theta + \gamma)$$

c $\tan.^{-1} a$ signifies a circular arc whose tangent $= a$. In the same way we use $\sin.^{-1} a$, &c.

THE CLASSES AND DEGREES OF EQUATIONS.

(9) Def. If an indeterminate equation consist of a *finite* series of terms, in which the variables x and y occur in their *whole positive powers*, or so involved that the equation is reducible to such a series, it is called distinctively an *algebraic equation*.

Any equation not of this kind is called a *transcendental equation*.

(10) The *degree* or *dimension* of an algebraic equation is the number equal to the exponent of the variable or sum of the exponents of the variables in that term of the equation in which it is the greatest; it being always presupposed that the equation is cleared of roots, and reduced to its simplest form.

In investigating the loci of equations, we class those which are algebraic according to their degrees. We shall consider these first, and those of the transcendental kind afterwards.

When we speak of an equation simply, it will be understood to mean an indeterminate equation, unless the contrary is specified.

DIVISION I.

ALGEBRAIC EQUATIONS.

SECTION I.

EQUATIONS OF THE FIRST DEGREE.

Investigation of the Locus.

(1) According to the Definition above, an equation of the first degree is one in which the power of each of the variables is unity.

An equation of the first degree may be represented in its most general and complete form thus:

$$Ay + Bx + C = 0,$$

the coefficients being supposed affected by their proper signs.

This form includes all possible cases: in fact, the only variation which could be made would be by supposing $C = 0$, in which case the equation would still be of the first degree. If either A or $B = 0$, it would become determinate; if both $= 0$, it would be reduced to $C = 0$, or no equation of the variables would remain. We shall first investigate the locus of the equation in its complete form, and then shew to what conditions a change in its form corresponds.

By transposing and dividing we obtain,

$$\frac{y + \frac{C}{A}}{x} = -\frac{B}{A}$$

Or, referring to coordinates, we have the ratio between the variable $y +$, a constant part, and the variable x, $=$ a given ratio.

(2.) Thus, Fig. (1) taking the coordinates xy, inclined at an angle ω, since y must be measured from the origin O, let OC be taken $= \frac{C}{A}$. Then measuring any simultaneous values of x and y from O, and drawing parallels, we have the locus P; and $\frac{Cy}{yp}$ = *a constant ratio.*

This therefore is the characteristic property of the locus of the first degree: of which it is evident there is but this one species.

Also the locus is of unlimited extent on both sides of the axis.

(3.) To find *to what geometrical conditions this locus corresponds.* From a fixed point, as C, draw Cy, and yp in the given ratio to it, at an angle $= \omega$. Take other successive values in the same manner; and it is evident, from the principles of geometry, that we have *a series of similar triangles; and the points* $pp_{,}$, *&c. will lie in a straight line passing through* C.

COR. Hence if in any different equations of the first degree we have the *ratio* $-\frac{B}{A}$ *the same,* the loci will be straight lines *parallel to each other.* For C and A having in the second equation different values, the point C must be taken at a different distance from the origin: and the corresponding values of the terms of the ratio Cy yp will be different, though the ratio continues the same. Hence the locus will be a straight line parallel to Cp, and situated on the same side of it as the new point C is with respect to the former point C.

(4.) Designating the line which is the locus of the equation by the symbol l and the $\angle$ it forms with x and y, $< lx < ly$; (the sides of a $\triangle$ being as the sines of the opposite angles;) we have,

$$\frac{B}{A} = \frac{\sin. lx}{\sin. ly}$$

If $B = 0$ $\sin. lx = 0$, or lx is parallel to X.

Also the equation becomes

$$y + \frac{C}{A} = 0,$$

which gives a determinate value of y, and the locus passes through the point B so determined.

In like manner if $A = 0$, the locus is parallel to the axis Y. Thus, in general, *if the coefficient of either variable* $= 0$, *the locus is parallel to the axis to which that variable is referred.*

(5.) If in either case we have also $C = 0$, then $Ay = 0$, or $Bx = 0$, or the locus *coincides* with the axis X or Y respectively.

(6.) If the other coefficients are finite, and $C = 0$, the points C and B coincide with O, or the locus passes through the origin.

(7.) To express the angle contained by two lines in terms of the coefficients of their equations and the angle of ordination.

Let the equations be

$ay + bx + c = 0$, giving the line l,

$\alpha y + \beta x + \gamma = 0$ . . . . . . λ.

From (4.) $\dfrac{\sin. lx}{\sin. ly} = -\dfrac{b}{a}$ and $\dfrac{\sin. \lambda x}{\sin. \lambda y} = -\dfrac{\beta}{\alpha}$

But $\angle\, ly = \omega - lx$ $\qquad \lambda y = \omega - \lambda x.$

$$\therefore \frac{\sin. lx}{\sin. \omega \cos. lx - \cos. \omega \sin. lx} = -\frac{b}{a}$$

$$=\frac{\frac{\sin. lx}{\cos. lx}}{\sin. \omega - \cos. \omega\, \frac{\sin. lx}{\cos. lx}}$$

$$=\frac{\tan. lx}{\sin. \omega - \cos. \omega \tan. lx}$$

$$\therefore \tan. lx = -\frac{b}{a}(\sin. \omega - \cos. \omega \tan. lx)$$

$$= \frac{-b \sin. \omega}{a} + \frac{b \cos. \omega \tan. lx}{a}$$

$$\tan. lx - \tan. lx \frac{b \cos. \omega}{a} = \frac{-b \sin. \omega.}{a}$$

Whence, $\tan. lx = \dfrac{-b \sin. \omega}{a - b \cos. \omega}$

And in a manner exactly similar,

$$\tan. \lambda x = \frac{-\beta \sin. \omega}{\alpha - \beta \cos. \omega}$$

But $\angle\, l\lambda = lx - \lambda x$

$$\therefore \tan. l\lambda = \tan. (lx - \lambda x)$$

$$= (\text{trigon.})\ \frac{\tan. lx - \tan. \lambda x}{1 + \tan. lx \tan. \lambda x}$$

Or substituting the values above,

$$= \frac{\dfrac{b \sin. \omega}{b \cos. \omega - a} - \dfrac{\beta \sin \omega}{\beta \cos. \omega - \alpha}}{1 + \dfrac{b \sin. \omega}{b \cos. \omega - a} \cdot \dfrac{\beta \sin. \omega}{\beta \cos. \omega - \alpha}}$$

$$= \frac{b\beta \cos. \omega \sin. \omega - \alpha b \sin. \omega - \beta b \sin. \omega \cos. \omega + a\beta \sin. \omega}{b\beta \cos.^2 \omega - \alpha b \cos. \omega - a\beta \cos. \omega + a\alpha + b\beta \sin.^2 \omega}$$

$$\text{or } \tan. l\lambda = \frac{(a\beta - \alpha b) \sin. \omega}{a\alpha + b\beta - (a\beta + \alpha b) \cos. \omega}.$$

Cor. Hence if $l\lambda = \frac{\pi}{2}$ $\tan. l\lambda = \infty$

$$\therefore a\alpha + b\beta - (a\beta + \alpha b) \cos. \omega = 0.$$

(8.) To express the equation to a straight line which shall pass through a given point, determined by the coordinates x_2 y_2.

Let the equation be $Ay + Bx + C = 0$,

And since the given point is by supposition in the line to which this equation belongs, the equation for that point becomes $Ay_2 + Bx_2 + C = 0$.

Subtracting this from the former we have $\left.\right\} A\,(y - y_2) + B\,(x - x_2) = 0$,

which is the form required.

(9.) To find the polar equation corresponding to an equation of coordinates of the first degree.

In the given equation $Ay + Bx + C = 0$, substitute the values of x and y from the forms (A) and (B) (Introd.) and we have,

$$Ar\,\frac{\sin.\,\theta}{\sin.\,\omega} + Br\,\frac{\sin.\,(\omega - \theta)}{\sin.\,\omega} + C = 0$$

$$\therefore r = \frac{-C\,\sin.\,\omega}{A\,\sin.\,\theta + B\,\sin.\,(\omega - \theta)}.$$

SECTION II.

EQUATIONS OF THE SECOND DEGREE.

§. 1. *Solution of the equation, and investigation of its locus.*

(1.) An equation of the second degree is, according to the definition, one in which the power, or sum of the powers, of the variables in some term or terms $=2$, and in none exceed it. The most general and complete form of such an equation would be one including every combination of the powers of the variables, subject to this condition, and would be expressed thus;

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

A general solution of this equation is readily obtained for either of the variables, in terms of the other and the constants. Taking it for y, we have by transposition,

$$Ay^2 + (Bx + D)\,y = -Cx^2 - Ex - F$$

$$y^2 + \frac{Bx+D}{A}y + \frac{(Bx+D)^2}{4A^2}$$

$$= \frac{(Bx+D)^2}{4A^2} - \frac{C}{A}x^2 - \frac{E}{A}x - \frac{F}{A}$$

$$\therefore y = -\frac{Bx+D}{2A}$$

$$\pm\sqrt{\left(\frac{B^2x^2 + 2BDx + D^2 - 4ACx^2 - 4AEx - 4AF.)}{4A^2}\right)}$$

$$\text{or } y = \frac{-Bx+D}{2A}$$

$$\pm\sqrt{\left(\left[\frac{B^2-4AC}{4A^2}\right]x^2 + \left[\frac{2BD-4AE}{4A^2}\right]x + \frac{D^2-4AF}{4A^2}\right)}$$

(2.) If we refer this equation to coordinates, it is obvious that the value of y will consist of the aggregate of two parts, corresponding to the two members of this expression; the first giving a locus determined by a part measured from the axis X, as m (Fig. 5.) the second a locus determined by a part measured from m in the same line; but having two equal values, one above, the other below, m, corresponding the signs $+$ and $-$, and therefore giving a locus pq, in two branches, one on each side of the locus m, at equal distances.

If the second member were to become $=0$, the first would give,

$$2Ay+Bx-D=0$$

which being of the first degree, *the locus* m *is a straight line.* It is called a *diameter:* and the lines as pmq, bisected by it, and all parallel to Y, are called its *ordinates.*

If we had solved the equation for x in terms of y, we should have had an expression analogous to the former, and should only trace the locus in the direction of the axis Y. In such a solution the first member would give another diameter, whose equation would be

$$2\,Cx+By-E$$

having its ordinates parallel to X.

The nature of the locus pq will thus depend essentially on the second member of the expression, which we proceed to examine.

Resuming the solution for y, it may for the sake of brevity be expressed thus;

$$y=k+mx\pm\sqrt{(px+qx^2+n)}$$

Now considering only the second member, or the *quantity under the radical sign,* and writing it $=r^2$ we have,

$$x^2+\frac{p}{q}x+\frac{n}{q}=\frac{r^2}{q}$$

This being a quadratic equation we may, by a property of equations, easily reduce it to a form which will give us the conditions on which its values depend.

In order to solve the equation we take

$$x^2+\frac{p}{q}x+\frac{p^2}{4q^2}=\frac{r^2}{q}-\frac{n}{q}+\frac{p^2}{4q^2}$$

$$\text{which we may write}=\frac{r^2}{q}+s \quad . \quad . \quad . \quad . \quad . \quad (A)$$

$$\text{whence } x=-\frac{p}{2q}\pm\sqrt{\left(\frac{r^2}{q}+s\right)}$$

Now if $r^2=0$, and x_2 x_3 be the values of x corresponding, we have

$$x_2=-\frac{p}{2q}+\sqrt{s}$$

$$x_3=-\frac{p}{2q}-\sqrt{s}$$

$$\text{whence, } x-x_2=x+\frac{p}{2q}-\sqrt{s}$$

$$x-x_3=x+\frac{p}{2q}+\sqrt{s}$$

Multiplying these expressions together, we obtain

$$(x-x_2).(x-x_3)=x^2+\frac{p}{q}x+\frac{p^2}{4q^2}-s$$

but this result is the same as the original equation (A)

$$\text{Or we have } (x-x_2).(x-x_3)=\frac{r^2}{q} \quad . \quad . \quad . \quad . \quad (B)$$

The equation being reduced to this form, we can readily investigate the different cases in which the values assigned to x will give r^2, real or impossible, or $=0$, on which conditions the nature and form of the locus depend.

It may first be observed that in this investigation we wrote

$$s = \frac{p^2}{4q^2} - \frac{n}{q}$$
$$= \frac{p^2 - 4nq}{4q^2}$$

and x_2 x_3 will therefore be real when $p^2 > 4nq$

. . . . impossible when $p^2 < 4nq$

When real, their values will be equal, if $s = 0$, or unequal if it have any other value.

These conditions depend wholly on the assumed constants.

We have also, from the expression above,

$$x_3 - x_2 = 2\sqrt{s} = 2\sqrt{\left(\frac{p^2 - 4nq}{4q^2}\right)}$$
$$= \frac{1}{q}\sqrt{(p^2 - 4nq)}$$

(5.) From considering the form

$$(x - x_2)\,(x - x_3) = \frac{r^2}{q}$$

We obviously see that the sign of $\frac{r^2}{q}$ will depend upon that of q; and that of the factors on the first side of the equation, which will arise from the relative magnitudes of x x_2 x_3.

These different cases may be best exhibited in the following tabular view: in which we first distinguish the values of x_2 and x_3, when real and unequal, real and equal, or impossible. And under each case take the two suppositions $+q$, $-q$.

If we have $q = 0$, r^2 ceases to give a quadratic equation, and we have

$$px + n = r^2$$

or $$x + \frac{n}{p} = \frac{r^2}{p}$$

and we have only one value x_2 corresponding to $r^2 = 0$, which is $x = -\frac{n}{p}$

			CASE I. $+q$.	CASE II. $-q$
x_2 & x_3 real.	Unequal $x_3 > x_2$	$x_3 > x > x_2$	In equation B we have the factors with the signs $(-)(+) = -\left(\frac{r^2}{q}\right)$ $\therefore -r^2$ (imp.) (1)	$\ldots -\left(\frac{r^2}{-q}\right)$ $\therefore +r^2$ (real) (1)
		$x > x_3 > x_2$, $x_3 > x_2 > x$	$\left.\begin{matrix}(+)(+)\\(-)(-)\end{matrix}\right\} = +\left(\frac{r^2}{q}\right)$ (real) (2)	$\ldots +\left(\frac{+}{-}\right) = -$ (imposs.) (2)
		$x = x_2$, $x = x_3$	$0 = \frac{r^2}{q}$ (=0) (3)	.... (=0) (3)
	Equal $x_3 = x_2$	$x = x_2 = x_3$	$0 = \frac{r^2}{q}$ (=0) (4)	.... (=0) (4)
		All other values of x	$\left.\begin{matrix}(+)(+)\\(-)(-)\end{matrix}\right\} = +$ ... real) (5)	$\ldots +\left(\frac{+}{-}\right)$ (imposs.) (5)
x_2 and x_3 impossible.		$-s$	$+\frac{r^2}{q}$ (real) (6)	$\ldots +\left(\frac{+}{-}\right)$ (imposs.) (6)

CASE III.

$q = 0.$

	$+p$	$-p$	$p=0$
$x = x_2$	$r^2=0$ (1)	.... $r^2=0$ (4)	$r^2 = \begin{cases} +n \text{ (real) (7)} \\ -n \text{ (imp.) (8)} \\ 0 \quad (9) \end{cases}$
$x > x_2$	$+\left(\frac{r^2}{p}\right) = +$ ... (real) (2)	$\ldots +\left(\frac{r^2}{-p}\right) = -$ (imposs.) (5)	
$x < x_2$	$-\left(\frac{r^2}{p}\right) = -$ (imposs.) (3)	$-\left(\frac{r^2}{-p}\right) = +$ (real) (6)	

To trace out the locus in these different cases.

CASE I. $+q$. (Fig. 5.)

(No. 1.) Measuring off from 0 on the axis X, parts $=x_2$ and x_3, if x be taken any where between these points, there is no locus p, q, corresponding to the second member of the equation.

(2) (3) If parallels to the axis Y be drawn through these points they are tangents to the locus at VV_2.

And beyond these parallels the locus extends indefinitely in two branches.

(4) If the parts $=x_2$ and x_3 coincide, the points VV_2 coincide.

(5) And (retaining this supposition) if the values of x be taken on either side of these points, the equation becomes

$$(x-x_2)^2\, q=r^2$$

whence $$y=k+mx\pm\sqrt{q}\,(x-x_2)$$

a form including two equations of the first degree. The locus therefore becomes two straight lines: and since the equation may be expressed thus,

$$y=k+(m+\sqrt{q})\,x-x_2\sqrt{q}$$
$$y=k+(m-\sqrt{q})\,x+x_2\sqrt{q}$$

in which the ratios of the coefficients of x and y are different; consequently the lines must intersect, (i. 3.) and when $x=x_2$, the expressions are reduced to one value of $y=k+mx_2$. Which gives the point of intersection.

(6.) If there be no point x corresponding to $r^2=0$, take x corresponding to the least value of r^2, that is, of $mp\ mq$. (Fig. 9.) Through pq conceive parallels to the diameter to pass. Between these there will be no locus. It will touch them at p, q, and extend indefinitely beyond them each way. Thus in this case (which cannot occur at the same time as the former) we have a similar locus.

Def. The locus of the second degree, characterized by $+q$, that is, $+(B^2-4AC)$ is called *an hyperbola*[c].

(7.) Case II. $-q$. (Fig. 6.)

(1) Making a similar construction to the last, the locus is included between the parallels.

(2) No part of it extends beyond them.

(3) They are tangents at VV_2.

(4) In this case the locus becomes a point.

(5) and (6) There is no locus.

Def. The locus of the second degree, characterized by $-q$, that is, $-(B^2-4AC)$ is called *an ellipse*.

(8.) Case III. $q=0$. (Fig. 7.)

(1) The locus touches the parallel through x_2 at V.

(2) It extends indefinitely beyond it on the positive side.

(3) On the negative side there is no locus.

(4) (5) (6) The conditions are exactly the same, but the whole lies on the opposite side of both axes.

(7) (8) The equation becomes

$$y = k + mx \pm \sqrt{n}$$

which includes two equations of the first degree, having the same ratio of the coefficients of x and y: and consequently representing two parallel straight lines.

(9) No locus.

Def. The locus of the second degree, characterized by $q=0$, that is, $B^2-4AC=0$, is called a *parabola*.

[c] The student will be careful not to attach any other meaning to these terms than what is strictly implied in the definition of them.

VARIETIES IN THE FORM OF THE EQUATION.

).) We have thus far conducted our investigation, ng the equation in its general form. We have now xamine the varieties of form of which it is suscep-, and how such changes will affect the locus.

; is evident that such changes of form will consist ome of the coefficients being supposed $=0$, or cer- terms to be wanting. At least one of the three terms must be present, in order that the equation continue to be of the second degree: and terms lving x and y, that it may continue to be indeter-ate.

Ve will first consider the changes of form as taking e by the deficiency of some of the three first terms. pon the three first coefficients the species of the s has been shewn essentially to depend; but no-g is assumed respecting them, except the designa- of the coefficient q, or (B^2-4AC) as $+$, $-$, or . The characteristics, therefore, of each species ain unaltered, whatever supposition is made re-ting the values of A, B, and C, *consistent with assumed designation of* q.

0.) 1st. The condition which gives the hyperbola, viz.

$$+(B^2-4AC)$$

remain unaltered on any of the following supposi-s;

l) $A=0$ (2) $C=0$ (3) $A=C=0$.

) $B=0$ and $\therefore$ A and C with different signs, so $-4AC$ may become $+$.

' we adopt successively these suppositions in the ral equation, we obtain these variations in its :

$$\left.\begin{array}{ll}(1) & \quad\quad Bxy + Cx^2 \\ (2) & \quad\quad Bxy \\ (3) & Ay^2 + Bxy \\ (4) & \pm Ay^2 \quad\quad \mp Cx^2\end{array}\right\} + Dy + Ex + F = 0.$$

(11.) Of these forms, however, it is essential to observe, that the first supposes $A=0$; in which case the equation cannot be solved for y, as above. In this case, however, we might solve it for x, and obtain exactly similar results.

In the second form, where we have both $A=0$ and $C=0$, we can give no solution of the equation as before, either for x or y. It may, however, easily be shewn, that in this case, by a mere transformation to *other axes*, (without affecting the nature of the locus,) we derive a form involving both y^2 and x^2 with finite coefficients. For this form $Bxy + Dy + Ex + F = 0$ is evidently equivalent to

$$\tfrac{1}{4}B\left((y-x)^2-(y+x)^2\right)+Dy+Ex+F=0.$$

Now let us suppose the locus referred by coordinates $x_2 y_2$ to rectangular axes; and the axis X_2 *bisecting* the angle, (which we will call 2ϕ,) formed by the oblique axes: we have,

$$\angle x_2 y = \phi, \; xx_2 = -\phi, \; yy_2 = \frac{\pi}{2} - \phi, \; xy_2 = \frac{\pi}{2} + \phi.$$

Hence to express the rectangular, in terms of the oblique coordinates, the formula (Introd. 6. Case 3.) gives us,

$$\begin{aligned} y_2 &= -x \sin.\phi + y \sin.\phi &= (y-x)\sin.\phi \\ x_2 &= x\cos.\phi + y\cos.\phi &= (y+x)\cos.\phi. \end{aligned}$$

Let us assume $\alpha = \dfrac{\frac{1}{4}B}{\sin.^2\phi}$, $\gamma = \dfrac{\frac{1}{4}B}{\cos.^2\phi}$

$$\therefore \tfrac{1}{4}B\,(y-x)^2 = \alpha y_2^2, \quad \tfrac{1}{4}B(y+x)^2 = \gamma x_2^2.$$

Again, let $D=k+g$. $E=k-g$,

And assume $\delta=\frac{g}{\sin.\phi}$, $\epsilon=\frac{k}{\cos.\phi}$,

Then $\delta y_2=(y-x)\,\delta\sin.\phi$

$\epsilon x_2=(y+x)\,\epsilon\cos.\phi$

$$\therefore \delta y_2+\epsilon x_2=gy-gx+ky+kx$$
$$=(k+g)\,y+(k-g)\,x.$$

Thus collecting these results together, the original form becomes

$$\alpha y_2^2-\gamma x_2^2+\delta y_2+\epsilon x_2+F=0.$$

which expresses the *same locus* referred to *rectangular axes* having the same origin, but so inclined to the former that *the angle XY is bisected* by the rectangular axis X; and this might (if it were worth while) be transformed to axes forming *any* angle, which would give the complete equation. Thus the second form is properly included under the general designation of the hyperbola.

(12.) 2dly. The condition which gives the ellipse

$$-(B^2-4AC)$$

will remain unaltered, only on the supposition $B=0$; for since B^2 is essentially positive, neither A nor C can $=0$; and when $B=0$, $4AC$ must be positive, in order that the negative sign may remain, or A and C must each have the same sign. Hence the only variation which the equation admits in this case is,

$$(5)\quad Ay^2+Cx^2+Dy+Ex+F=0.$$

(13.) 3dly. The condition which gives the parabola

$$(B^2-4AC)=0,$$

will remain unaltered if we have (1) $B=0$ and $A=0$, or (2) $B=0$ and $C=0$.

Hence the variations of the general equation are,

$$\begin{matrix}(6)\ Ay^2 \\ (7)\ Cx^2\end{matrix}\Bigg\} + Dy + Ex + F = 0.$$

In the latter case the equation must be solved for x, as before remarked.

Of these forms (5) and (4) differ only in their signs: and it is evident that they include all the possible varieties, supposing the latter part of the general formula to remain unaltered.

(14.) The changes in this latter part, corresponding respectively to the deficiency of each of the three last terms, when combined either all together, each two together, or each singly, with each of the variations before investigated, give the whole of the variations in form of which the equation is susceptible. And these conditions $D=0$, $E=0$, $F=0$, will not affect the *species* of the locus, since the solution of the equation admits of k, m, n, and p, being either $+$, $-$ or $=0$.

Hence it is evident that *any indeterminate equation whatever of the second degree must correspond to some case of the three curves* above investigated; and they are therefore the *only species* of the locus of the second order.

We have now to shew to what conditions in the locus the deficiency of each of the last coefficients will correspond.

(15.) 1st. If the locus pass through the origin, whatever be its nature, we must have such an equation that when $x=0$, $y=0$; and consequently the general equation is reduced to . . . $F=0$.

The deficiency therefore of F in the general equation corresponds to this condition.

2d. The equation of the diameter, whose ordinates are parallel to the axis Y, is (2).

$$2Ay + Bx + D = 0.$$

If this diameter *pass through the origin*, we have (I. 4.) . . . $D=0$. If at the same time $B=0$, it *coincides* with X. (I. 5.)

3d. The equation of the diameter, whose ordinates are parallel to the axis X, is (2).

$$2\,Cx + By + E = 0.$$

If this diameter *pass through the origin*, we have (I. 4.) . . . . $E=0$.

And these conditions holding good conversely, if in the equation these several coefficients $=0$, the locus is in no other way affected than in its *position.*

(16.) It appeared before, that in all the cases we might have $B=0$ without affecting the species of the locus; and we may also have $F=0$, together with $D=0$, and still retain the three species. These assumptions therefore may be made, and the form still be general for all the curves.

Therefore assuming the equation with $B=0$, $D=0$, $F=0$, we have the *origin on the curve*, and the *diameter* to which it is referred *coinciding* with the *axis* X, and having its ordinates parallel to Y; with which the *tangent* at the vertex *coincides*. By adopting these suppositions we shall greatly simplify the subsequent investigation.

With these conditions then, the general expression is reduced to

$$Ay^2 + Cx^2 + Ex = 0.$$

k becomes$=0$, $m=0$, and $n=0$.

and the coefficients p and q become

$$p_2 = -\frac{E}{A},\quad q_2 = -\frac{C}{A}.$$

or writing these values of the coefficients, p and q,

$$y = \pm\sqrt{(px + qx^2)}.$$

Also the distance VV_2 becomes $= \frac{p}{q}$.

§. 2.

GENERAL PROPERTIES OF THE CURVES OF THE SECOND DEGREE.

DIAMETERS.

(17.) We have seen that in every case the locus of the second degree has a diameter bisecting a system of parallel chords. It becomes a question whether other systems of parallel chords may not in like manner have a diameter bisecting them; or the problem will be *to find the locus of the points of bisection of a system of parallel chords.*

(Fig. 8.) Assume as the equation of any chord pq,

$$ay + bx + c = 0.$$

Whence $y = -\frac{b}{a}x - \frac{c}{a} = -\frac{(bx + c)}{a}$

Let the ratio $\frac{b}{a}$ be constant for all the chords, $\frac{c}{a}$ being susceptible of different values peculiar to each chord: the equation then represents all the system of parallel chords. Substituting this value of y in the general equation (16) it becomes,

$$A\left(\frac{-bx - c}{a}\right)^2 + Cx^2 + Ex = 0$$

Whence expanding and multiplying by a^2 we get

$$(Ab^2 + Ca^2)x^2 + \left(Ea^2 + 2Abc\right)x + Ac^2 = 0$$

$$\therefore \quad x^2 + \left(\frac{Ea^2 + 2Abc}{Ab^2 + Ca^2}\right) x + \frac{Ac^2}{Ab^2 + Ca^2} = 0$$

which may be written for brevity

$$x^2 + hx + k = 0;$$

Whence $x = -\frac{h}{2} \pm \frac{1}{2}\sqrt{(h^2 - 4k)}$

Or writing the two values for the points p, q, x_p and x_q, and that for m, the point of bisection, x_m we have,

$$x_m = \frac{x_p + x_q}{2} = \frac{-h}{2}.$$

But $c = -ay - bx$.

Substituting this value in the expression which we have represented by h, we have

$$x_m = \frac{-[2Ab(-ay - bx)] - Ea^2}{2(Ab^2 + Ca^2)}.$$

Whence multiplying and collecting terms, (since x and y now belong to the point m,) we have

$$2\,Ca^2x_m = -Ea^2 + 2Abay_m$$

$$\text{or, } 2Cax_m - 2Aby_m + Ea = 0,$$

which is the *equation for the locus of bisection of the parallel chords:* and this being of the first degree, the *locus is a straight line:* and the equation of the parallel chords being arbitrarily assumed and applying to any system whatever, it follows that *in all curves of the second degree every system of parallel chords has a diameter bisecting them.*

(18.) If the locus be a parabola we have $C=0$, which being the coefficient of the term involving x, it follows from (i. 3.) that the equation which thus becomes

$$-2Aby+Ea=0$$

will represent a straight line *parallel to the axis X*, to which the diameter at first taken was supposed parallel: and this equation belonging to any diameter whatever, *in the parabola all diameters are parallel.*

(19.) In the other species of curves this condition does not hold good: and consequently in them none of the diameters represented by the above equation can be parallel to that which we have supposed coinciding with the axis X.

To find the point at which any diameter meets the axis X we have only to observe, that when this takes place the ordinate y of the diameter becomes $=0$, and we have

$$2\,Cax+Ea=0,$$

$$\text{or } x=\frac{-E}{2C},$$

an expression which is independent of the terms peculiar to any particular diameter; consequently *all diameters intersect each other in the same point* belonging to this value of x: this point is termed the *centre*. And referring to the general equation of the curves, we observe that the distance between the vertices or $VV_{,}=\frac{p}{q}=\frac{E}{C}$, consequently the *centre*, as above determined, is the *point of bisection* of the *diameter* coinciding with the axis Y. This value in the parabola when $C=0$ is infinite.

CONJUGATE DIAMETERS.

(20.) In the equation to the *ordinates* of any diameter the ratio of the coefficients of x and y, or $\frac{b}{a}$ is constant, since the ordinates are parallel.

Let the corresponding ratio in the equation to the ordinates of any other diameter be $\frac{\beta}{\alpha}$.

In the central curves let the *first diameter be parallel to the ordinates of the second:* then we have the ratio of the corresponding coefficients of the *first diameter* equal to that of the *second ordinates,* or

$$\frac{2Ca}{-2Ab}=\frac{\beta}{\alpha}.$$

Whence we have

$$\frac{2C\alpha}{-2A\beta}=\frac{b}{a}.$$

Or the *second diameter is parallel to the ordinates of the first.* Such diameters are called *conjugate diameters;* and the equation being general, the number of pairs of such diameters is unlimited.

(21.) This investigation applies wherever we can assume that a diameter may be parallel to the ordinates of a second.

In the ellipse this is evidently the case with any diameter, and therefore every diameter has a conjugate; and this being one of its ordinates, and meeting it in the centre, is there bisected. Hence *every diameter is bisected in the centre.*

In the hyperbola the above assumption can never be made; since the ordinates to any diameter being parallel to the tangents at its vertices, a line through the centre parallel to these tangents can never meet either of the opposite curves, or be a conjugate diameter in the sense above defined.

(22.) To express the *angle* γ, formed by the two *conjugate diameters*, in terms of the constants of their equations.

The equation of the first diameter being (17)

$$2Aby - 2Cax + Ea = 0.$$

And that of its conjugate included under the ordinates,

$$ay + bx + c = 0.$$

From (I. 7.) we have, substituting these values and dividing every term by 2,

$$\tan.\ \gamma = \frac{(Ab^2 - Ca^2)\ \sin.\ \omega.}{Aba + Cab - (Ab^2 + Ca^2)\cos.\ \omega}.$$

To find whether in any and in what cases two *conjugate* diameters can be *at right angles*.

If the *conjugate* diameters be at right angles, we have from the above expression, by (Cor. I. 7.)

$$Aba + Cab - (Ab^2 + Ca^2)\ \cos.\ \omega = 0\,;$$

or dividing by Aa^2

$$\frac{b}{a} + \frac{Cb}{Aa} - \left(\frac{b^2}{a^2} + \frac{C}{A}\right)\ \cos.\ \omega = 0.$$

Whence transposing and dividing by cos. ω

$$\frac{b^2}{a^2} + \left(\frac{A+C}{A\ \cos.\ \omega}\right)\frac{b}{a} = -\frac{C}{A}$$

$$\text{Or } \frac{b}{a} = -\frac{A+C}{2A\ \cos.\ \omega} \pm \sqrt{\left(-\frac{C}{A} + \frac{(A+C)^2}{4A^2\ \cos.^2\omega}\right)}$$

Now if the equation had been solved for $\frac{Ca}{Ab}$, we should have (dividing by Ab^2)

$$\frac{a}{b}+\frac{Ca}{Ab}-\left(1+\frac{Ca^2}{Ab^2}\right)\cos.\,\omega=0$$

whence $$\frac{Ca^2}{Ab^2}-\left(\frac{A+C}{A\cos.\,\omega}\right)\frac{a}{b}=-1$$

or multiplying by $\frac{C}{A}$

$$\frac{C^2a^2}{A^2b^2}-\left(\frac{A+C}{A\cos.\,\omega}\right)\frac{aC}{bA}=-\frac{C}{A}$$

whence $$\frac{Ca}{Ab}=-\frac{A+C}{2A\cos.\,\omega}\pm\sqrt{\left(-\frac{C}{A}+\frac{(A+C)^2}{4A^2\cos.^2\omega}\right)}$$

Which is the same expression as that we obtained for $\frac{b}{a}$. But the expression has two values. These two values therefore are respectively equal to the two ratios of the coefficients of the diameter and its conjugate, which are at right angles: and these two values are in terms of the original constants of the equation to the curve only; consequently in each curve there is one pair of conjugate diameters at right angles to each other, and only one. These are called the *principal diameters*, or (in geometrical investigations) *the axes* of the curve.

(23.) The general equation, (q remaining to be affected by its proper sign,) the *origin* being at the *vertex*, is,

$$y^2=px+qx^2.$$

which in the parabola becomes

$$y^2=px.$$

The constant p is called the parameter.

In the central curves this general expression is evidently $= q\left(\frac{p}{q}x + x^2\right)$

and (16) writing $\frac{p}{q} = 2a$, whence $q = \frac{p}{2a}$, it becomes

$$y^2 = \frac{p}{2a}\left(2ax + x^2\right)$$

$$\text{or } y^2 = \frac{p}{2a}\left(2ax \pm x^2\right) \left\{\begin{matrix}\text{Hyp.}\\ \text{Ell.}\end{matrix}\right\} \quad . \quad . \quad . \quad (C)$$

To find the form of the equation when the *origin* is transferred to the *centre*, we have only to observe that the second member of the general expression above may be written thus,

$$a^2 - a^2 + 2ax + x^2$$

or we have,

$$y^2 = \frac{p}{2a}\left(a^2 - (a + x)^2\right)$$

And the abscissa, reckoned from the *centre*, becomes expressed generally $(a + x)$, or $(a \pm x)$ $\left\{\begin{matrix}\text{Hyp.}\\ \text{Ell.}\end{matrix}\right.$

Whence, now writing x for the new abscissa, we have in general,

$$y^2 = \frac{p}{2a}(a^2 - x^2)$$

$$\text{or } y^2 = \frac{p}{2a}\left(\mp a^2 \pm x^2\right) \left\{\begin{matrix}\text{Hyp.}\\ \text{Ell.}\end{matrix}\right\} \quad . \quad . \quad . \quad (D)$$

(24.) The general equation $y^2 = px + qx^2$ corresponds to the case of a diameter coinciding with the axis X, and having in all cases its vertical tangent coinciding with Y, and its ordinates parallel to Y: and in the ellipse its conjugate diameter also parallel to Y. And since the axes may be assumed with any angle of ordination, this equation applies to the curve as referred to any pair of conjugate diameters.

With the origin at the centre, and when $x=0$, we have

$$y^2 = \pm\frac{p}{2a}(a^2) = \pm\frac{pa}{2},$$

or writing this $= b^2$, we have

$$\frac{p}{2} = \frac{b^2}{a}.$$

But in the ellipse, in this case, y becomes the semi-conjugate diameter; hence *the parameter is a third proportional to the conjugate diameters.* And since the equation applies in the hyperbola also, we may have lines through the centre, parallel to the ordinates of any diameters, and determined in length by this proportion, which may be considered as conjugate diameters.

This equation gives us the property that the *rectangles of the abscissæ*, or in the parabola *the abscissæ simply*, are as the squares of the ordinates. Also in the parabola the parameter is a third proportional to any abscissa and its ordinate.

From the above expressions we have

$$\frac{p}{2a} = \frac{b^2}{a^2}.$$

And substituting this value the equations become

$$y^2 = \frac{b^2}{a^2}(2ax \pm x^2) \quad . \; . \; . \; . \quad (c)$$

$$y^2 = \frac{b^2}{a^2}(\mp a^2 \pm x^2) \quad . \; . \; . \; . \quad (d)$$

By these properties we are enabled to *identify* the *algebraic loci* of the second degree with the *geometrical curves* derived from the section of the *cone.*

(25.) In the ellipse and hyperbola it is evident that we may have $A = C$: in which case $q = 1$, and $\frac{p}{q} = 2a = p$.

$\therefore\ a = b$, and the equation above becomes

in the ellipse $y^2 = a^2 - x^2$

in the hyperbola $y^2 = x^2 - a^2 = (a^2 - x^2) \cdot (-1)$

$$\text{or } y = \sqrt{(a^2 - x^2)}\sqrt{-1}$$

This variety of the equation corresponds to the case of the equilateral hyperbola, and in the ellipse to the circle for all its diameters, and to the case of the pair of conjugate diameters, which are equal in every species of ellipse.

INTERSECTIONS OF DIAMETERS WITH THE CURVE:— AND ASYMPTOTES.

(26.) In the central curves to find the coordinates of the points at which any diameter meets the curve.

At this point the coordinates of the diameter will be identical with those of the curve. Taking the origin at the centre, (through which all diameters pass,) let the equation to the diameter be

$$\alpha y + \beta x = 0$$

$$\text{Whence } y = -\frac{\beta}{\alpha} x, \quad x = -\frac{\alpha}{\beta} y;$$

or writing $-\frac{\beta}{\alpha} = m$

$$y^2 = m^2 x^2, \quad x^2 = \frac{y^2}{m^2}.$$

The equation to the curve (d) gives

$$y^2 = \mp b^2 \pm \frac{b^2}{a^2} x^2,$$

$$\text{or } a^2y^2 \mp b^2x^2 = \mp a^2b^2 \begin{cases}\text{Hyp.}\\ \text{Ell.}\end{cases}$$

Hence, substituting for y we have,

$$a^2m^2x^2 \mp b^2x^2 = \mp a^2b^2,$$

$$\text{or } x^2 = \frac{\mp a^2b^2}{a^2m^2 \mp b^2}$$

and substituting for x,

$$a^2y^2 \mp \frac{b^2}{m^2} y^2 = \mp a^2b^2$$

$$\text{or } m^2a^2y^2 \mp b^2y^2 = \mp m^2a^2b^2$$

$$\therefore\ y^2 = \frac{\mp m^2a^2b^2}{a^2m^2 \mp b^2}.$$

Hence in the ellipse

$$x = \frac{ab}{\sqrt{(a^2m^2 + b^2)}}$$

$$y = \frac{mab}{\sqrt{(a^2m^2 + b^2)}}.$$

In the hyperbola (multiplying both terms of the fraction by -1) we have

$$x = \frac{ab}{\sqrt{(b^2 - a^2m^2)}}$$

$$y = \frac{mab}{\sqrt{(b^2 - a^2m^2)}}.$$

These values in the ellipse are always real; or every diameter meets the curve.

(27.) In the hyperbola the designation of these values will depend upon the value of b^2 compared with a^2m^2, and according as we have

$$b^2 \left\{ \begin{matrix} > \\ < \\ = \end{matrix} \right\} a^2m^2, \text{ or } \frac{b}{a} \left\{ \begin{matrix} > \\ < \\ = \end{matrix} \right\} m, \begin{matrix} x, \text{ and } y \\ \text{will be} \end{matrix} \left\{ \begin{matrix} \text{real} \\ \text{impossible} \\ = \infty \end{matrix} \right.$$

In the first case, the diameter meets each of the opposite curves to which the equation belongs.

In the second, it never meets them.

In the third case, or if we have $\frac{b}{a} = m$, the diameter only meets the curve in a point infinitely distant; or in other words, if in the hyperbola the equation to a line through the centre be

$$\alpha y + \beta x = 0$$

with the condition $\frac{\beta}{\alpha} = \frac{b}{a}$,

that line never meets the curve, or ceases to be a diameter, according to the definition, and *becomes an asymptote.*

(28.) From the equation to the asymptote just expressed, and writing the angle of inclination to the principal diameter $= \phi$, we have

$$\frac{y}{x} = -\frac{\beta}{\alpha} = \frac{b}{a} = \frac{\sin. \phi}{\cos. \phi}.$$

And when the ordinate to the asymptote meets the principal diameter at the vertex

$$x = a \quad \therefore y = b,$$

here $a = r. \cos. \phi$, $\quad b = r. \sin. \phi$, $\quad a^2 + b^2 = r^2$

(29.) We have already seen that the case of the general equation, involving the product xy, was, by a transference to other axes, brought into a form involving the squares of the variables. It will be easily seen that these axes are the asymptotes; and we thus have the equation of the same curve referred to the asymptotes as axes; or, as it is usually called, the *equation to the hyperbola between its asymptotes.*

When the form of the equation is that belonging to rectangular axes, and the origin at the centre, the corresponding form of the equation, referring to the asymptotes, is easily exhibited.

The equation in this case,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

is evidently of the form

$$\alpha y^2 - \gamma x^2 + \zeta = 0,$$

which by reversing the operation before (11), gives the form referring to the asymptotes,

$$\beta x_2 y_2 + \zeta = 0,$$

$$\text{or } x_2 y_2 = -\frac{\zeta}{\beta};$$

here $\zeta = b^2 a^2$, and $a^2 = \alpha = \frac{\frac{1}{4}\beta}{\sin.^2\phi} \therefore \beta = 4a^2 \sin.^2\phi$;

$$\text{whence } \frac{\zeta}{\beta} = \frac{b^2}{4\sin.^2\phi}.$$

But substituting the value of b^2 (28) this becomes

$$x_2 y_2 = \frac{\zeta}{\beta} = \frac{r^2 = a^2 + b^2}{4}$$

which is the form required.

§. 3.

FOCAL CONSTRUCTION; AND POLAR EQUATIONS.

(30.) In each of the curves let c = the distance measured from the vertex, at which the ordinate = the semi-parameter, meets the axis. The point at which this takes place is called *the focus*. We have then to find from the equations the value of the abscissa corresponding to this point.

In the parabola, since $x_f = c$, $y_f = \frac{p}{2}$,

$$\text{we have } \frac{p^2}{4} = pc, \text{ or } c = \frac{p}{4}.$$

Hence there is one such point at this distance from the vertex.

In the ellipse and hyperbola, (the centre being the origin,)

$$y_f = \frac{b^2}{a} \therefore y_f^2 = \frac{b^4}{a^2}.$$

Whence substituting in the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\text{and dividing by } \frac{b^2}{a^2}, \text{ we have}$$

$$b^2 = a^2 - x_f^2$$

$$\text{or } x_f = \pm\sqrt{(a^2 - b^2)}$$

Hence there are in the ellipse and hyperbola two such points at equal distances, one on each side of the centre.

The distance $a \pm c \left\{ \begin{matrix} \text{Hyp.} \\ \text{Ell.} \end{matrix} \right\}$ is called *the excentricity*.

And the ratio $\frac{a \pm c}{a} = e$, *the ratio of excentricity.*

This ratio $= 1 \pm \frac{c}{a}$, and hence we have

$$\left.\begin{array}{r}\text{In the parabola}\\ \text{hyperbola}\\ \text{ellipse}\end{array}\right\} e \left\{\begin{array}{c} = \\ > \\ < \end{array}\right\} 1$$

$$\text{Hence } \frac{b^2}{a^2} = 1 - \frac{x_f^2}{a^2} = 1 - \frac{(a \pm c)^2}{a^2} = 1 - e^2$$

And from the above values of e we have $\mp (1 - e^2) \left\{ \begin{array}{l}\text{Hyp.}\\ \text{Ell.}\end{array}\right.$

Hence also $ae = a \pm c$, or $a . \mp (1 - e) = \mp c \therefore a = \frac{c}{1 - e}$

(31.) Hence, by substitution in the general equation to all the curves, the origin being at the vertex, we obtain

$$y^2 = \mp (1 - e^2) \left(\frac{2cx}{1-e} \pm x^2\right) \left\{ \begin{array}{l}\text{Hyp.}\\ \text{Ell.}\end{array}\right.$$

$$\therefore y^2 = -x^2 + e^2x^2 + 2cx(1 + e)$$

$$y^2 + x^2 - 2cx + c^2 = e^2x^2 + 2cex + c^2$$

$$y^2 + (x - c)^2 = e^2 \left(\frac{c^2}{e^2} + \frac{2cx}{e} + x^2\right) = e^2\left(\frac{c}{e} + x\right)^2$$

Hence we derive the following construction;

(Conic Sect. Fig. 12, 13, 14.) Let VF be the abscissa $= c$

$$x = VM \quad y = PM$$

$$\therefore FP^2 = y^2 + (x - c)^2 \quad \therefore = e^2\left(\frac{c}{e} + x\right)^2$$

$$\therefore \frac{FP}{\frac{c}{e} + x} = e$$

$$\text{Take } \frac{V\Delta}{VF} = \frac{1}{e} \therefore V\Delta = \frac{c}{e}$$

$$\therefore \frac{c}{e} + x = V\Delta + VM = \Delta M$$

And we have $\frac{FP}{\Delta M} = e.$

or drawing the perpendicular $\Delta\Delta$ and $P\Delta$ at right angles to it, and which therefore $= \Delta M$, we have the *construction of the curves by the directrix:* or we might thus identify the loci of the second degree with the geometrical curves thus described in plano.

To find the polar equations to the curves of the second degree, the focus being the pole.

1st. In the parabola, the vertex being the origin, we have

$$y^2 = px = 4\,cx$$

And if the origin be transferred to the focus, the abscissa becomes $x - c = r \cos.\,\theta$. (Int. 7.)

$$\text{and } y = r \sin.\,\theta$$

$$\text{hence } r^2 = r^2(\sin.^2\theta + \cos.^2\theta) = x^2 - 2\,cx + c^2 + 4\,cx$$

$$= (x+c)^2$$

$$\therefore r = x + c = x - c + 2c = 2c - (c - x)$$

$$= \frac{p}{2} - r \cos.\,\theta$$

$$\therefore r = \frac{p}{2} \cdot \frac{1}{1 + \cos.\,\theta}$$

(33.) 2dly. In the ellipse and hyperbola, the centre being the origin, we have, substituting in the form (24. d.) the value given in (30.)

$$y^2 = (1 - e^2)\,(a^2 - x^2)$$

If the origin be transferred to the focus, the abscissa becomes $ae - x$, and taking the focus as the pole,

$$ae - x = r \cos.\,\theta \text{ by (Int. 7.)}$$

$$\text{or } x = ae + r \cos.\,\theta$$

$$\text{and } y = r \sin.\,\theta$$

Substituting these values in the equation above, we obtain

$$r^2 \sin.^2 \theta = (1-e^2)\ \{a^2-(ae+r\cos.\theta)^2\}$$

$$r^2 = \frac{1}{\sin.^2\theta}\left\{(1-e^2)\left(a^2-(a^2e^2+2\,aer\cos.\theta+r^2\cos.^2\theta)\right)\right\}$$

$$\text{or } r^2\left(\frac{\sin.^2\theta+(1-e^2)\cos.^2\theta}{\sin.^2\theta}\right)+r\left(\frac{2\,(1-e^2)\,ae\cos.\theta}{\sin.^2\theta}\right)$$

$$=\frac{(1-e^2)\ (a^2-a^2e^2)}{\sin.^2\theta}$$

$$r^2+r\frac{2(1-e^2)\,ae\cos.\theta}{1-e^2\cos.^2\theta}=\frac{(1-e^2)\,(a^2-a^2e^2)=a^2\,(1-e^2)^2}{1-e^2\cos.^2\theta}$$

Whence, solving the quadratic,

$$r=-\frac{(1-e^2)\ ae\cos.\theta}{1-e^2\cos.^2\theta}\pm\sqrt{\left\{\frac{a^2\,(1-e^2)^2}{1-e^2\cos.^2\theta}\right.}$$

$$\left.+\frac{(1-e^2)^2a^2e^2\cos.^2\theta}{(1-e^2\cos.^2\theta)^2}\right\}$$

$$r=\frac{1}{1-e^2\cos.^2\theta}\Big\{-(1-e^2)\ ae\cos.\theta$$

$$\pm\sqrt{\{a^2\,(1-e^2)^2\,(1-e^2\cos.^2\theta)+(1-e^2)^2a^2e^2\cos.^2\theta\}}$$

$$r=\frac{1}{(1-e\cos.\theta.)\ (1+e\cos.\theta.)}\Big\{-(1-e^2)ae\cos.\theta$$

$$\pm\sqrt{\{a^2\,(1-e^2)^2\}}\Big\}$$

$$r=\frac{1}{(1-e\cos.\theta)\,(1+e\cos.\theta)}\left[a\,(1-e^2)\ (\pm 1-e\cos.\theta)\right]$$

$$r=\pm\frac{a\,(1-e^2)}{1+e\cos.\theta}=\pm a\frac{b^2}{a^2}.\ \frac{1}{1+e\cos.\theta}=\pm\frac{p}{2}.\ \frac{1}{1+e\cos.\theta}$$

This expression includes that for the parabola where $e=1$.

(34.) To find the polar equation to the ellipse and hyperbola, the *centre* being the *pole*.

We have evidently

$$r^2=x^2+y^2\quad x=r\cos.\theta.$$

and substituting for y^2, this becomes

$$r^2 = x^2 + (1-e^2)(a^2 - x^2)$$
$$= a^2(1-e^2) + e^2x^2$$
$$= a^2(1-e^2) + e^2r^2 \cos.^2\theta.$$
$$\therefore r^2(1 - e^2\cos.^2\theta) = a^2(1-e^2)$$
$$\text{or } r = a\sqrt{\left(\frac{(1-e^2)}{1-e^2\cos.^2\theta}\right)}$$

This expression is general for both curves. In the hyperbola we have only to change the signs, since $e^2 > 1$.

(35.) In the circle $e = 0$: hence either of these polar equations gives $r =$ a constant quantity.

We have already seen that the most general form of the equation of coordinates to the circle was included under that of the ellipse when $A = C$. The constancy of the radius enables us to put that most general expression into a form which shews the value of the coefficients.

If we conceive a circle referred to axes forming any oblique angle ω, and having the coordinates of its centre given, α, β; the coordinates to any point in the curve being x, y, it is evident that joining this point with the centre an oblique triangle will be formed, of which r will be the side opposite to the angle$=\omega$ (or $\pi-\omega$); and the other two sides will be respectively $\alpha-x$ $\beta-y$. Also, if from the extremity of r a perpendicular be dropped on the opposite side (a) the portion intercepted from ω will be

$$(\beta - y)\cos.\ \omega.$$

Hence, by Euc. II. 12, we have

$$r^2 = (\alpha - x)^2 + (\beta - y)^2 + 2(\alpha - x)(\beta - y)\cos.\ \omega.$$

which, by multiplying and collecting terms, gives

$$\left.\begin{array}{l} y^2+x^2+2xy\cos.\omega. \\ -2(\beta+\alpha\cos.\omega)y \\ -2(\alpha+\beta\cos.\omega)x \\ +\beta^2+\alpha^2+2\alpha\beta\cos.\omega-r^2 \end{array}\right\}=0$$

an equation which is of the form of the general equation, and in which $A=C=1$. $B=2\cos.\omega$, &c.

If the axes be rectangular, $B=0$.

Upon the principles thus deduced a complete system of the properties of the curves of the second degree might be founded: and this has been done by several writers; as Lardner in his Algebraic Geometry, and Hamilton in his Analytic Geometry; to which works the reader is referred who is desirous of prosecuting the subject upon these principles. We have here carried the deductions only so far as to identify the algebraic loci of the second degree with the geometrical curves, whether considered as derived from the cone or described in plano, and to exhibit their polar equations. These afford the data for the fluxional investigation of their further properties and analogies.

SECTION III.

EQUATIONS OF THE THIRD DEGREE.

(1.) A general formula, exhibiting an equation of the third degree with all its terms, might easily be drawn out, in a manner similar to that in which we have given the equation of the second degree. It might also be shewn, in the same way, how many varieties of form it admits, and the particular cases of its locus might be classed and described. This has in fact been done by Newton, in his tract entitled *Enumeratio Linearum tertii Ordinis:* in which he shews that the equation may be ultimately reduced to four general forms. Of the first form alone he enumerates not less than sixty-five species, to which eight have since been added. The whole number of species of all the forms are perhaps yet uninvestigated. And the fact is, that in this (and indeed in the orders above the second generally) such investigation is immensely laborious, and of little or no use; a very few of the curves only having any application in other parts of mathematical or physical science.

The investigation of the equation of the second degree has been fully entered into, because several important conclusions could not have been derived without doing so: and such an investigation will sufficiently illustrate the most general discussion of an algebraic equation and its locus.

With respect, therefore, to the equation of the third degree, we shall not attempt any such general investigation, but shall merely select a few instances calculated to exemplify the nature of its loci, and to exhibit the primary properties of such of the curves as have any remarkable applications.

(2.) Let it be required to investigate the locus of the cubic equation $ax=y^3$; a being a constant quantity, supposed always positive, the axes rectangular, and the origin upon the curve.

The equation is reducible to the two following cases;

If we have $+x$ it gives $+y^3$, the root of which is $+y$:

. . . . $-x$ . . . . $-y^3$ . . . . $-y$:

Here y being in each case possible, and having the same sign as x, the locus will be constructed as in (Fig. 10.) or there will be two infinite branches lying on opposite sides of the axes X and Y.

This curve is called the *cubical parabola.* The investigation may be compared with that of the common parabola, where we have seen $-x$ gives y impossible, and $+x$ corresponds to two values $\pm y$.

(3.) In the same way, if we had the equation

$$ax^2=y^3$$

$+x$ gives $+ax^2$ and $\therefore$ $+y^3$, whose root $+y$ is real.

$-x$ . . $+ax^2$ . . . . $+y^3$ . . . . . . $+y$ . . . .

or the locus (as in Fig. 11.) will consist of two infinite arcs on the same side of X, but on opposite sides of Y. This curve is called the *semicubical parabola.*

(4.) Let the equation be $yx^2=a^3$

$$\text{or } y=\frac{a^3}{x^2}$$

Here a being supposed positive,

If we have $+x$ it gives $+x^2$, $\therefore\ y=+\frac{a^3}{x^2}$

$$-x\ \ldots\ldots\ +x^2\ \ldots\ y=+\frac{a^3}{x^2}$$

$$\pm x=0 \qquad y=\infty,\text{ and } y=0, \quad x=\infty.$$

Hence the locus lies on the same side of the axis X, but on opposite sides of Y; and the axes are asymptotes. (Fig. 12.)

(5.) With respect to these, and indeed all subsequent curves, it may be proper here to remark, that it is perfectly arbitrary which of the coordinates we designate as x, and which as y. Thus the curves are exactly the same as the two above, which are designated by the equations

$$ay=x^3, \text{ and } ay^2=x^3.$$

In tracing the locus, we should only have to reverse the position of the ordinate and abscissa.

We may also remark that these equations often occur in a less simple form than as here given; i. e. involving constants which determine the *position* of the locus. Thus the semi-cubical parabola is often met with under this form,

$$ny^2=(x-\frac{p}{2})^3$$

in which case it is to be observed, that, at the origin, when $y=0$, we have $x-\frac{p}{2}=0$, or $x=\frac{p}{2}$. The vertex of the curve $\therefore$ is at the distance $=\frac{p}{2}$ from the origin.

Here also the designation of the coordinates is reversed.

(6.) Let the given equation be

$$a^2y - x^2y - a^3 = 0.$$

This being solved for y, we obtain

$$y = \frac{a^3}{a^2 - x^2}.$$

Hence we have the following conditions:

If $x = \pm a$, $y = \infty$.

If $\pm x < \pm a$ $\left\{\begin{array}{l}\text{the denominator has the same sign as}\\ \text{the numerator, and we have } +y.\end{array}\right.$

If $x = 0$, $y = a$, $=$its minimum value.

If $\pm x > \pm a$ we have $-y$,

and as x increases, the denominator increases, and $\therefore$ y decreases; and this without limit. Hence the geometrical character of the locus will be as represented in (Fig. 13.) For portions being measured off from O, $= +a$, and $-a$ and parallels passed through them, those parallels will be asymptotes to the curve, since at those distances $y = \infty$.

Within these limits the curve lies wholly on one side of the axis X.

If a portion of Y be taken $= a$, this will be the vertex, and a parallel through it will be a tangent.

For values of x beyond the limits of the asymptotes, the curve lies below X; and this axis and the parallels become asymptotes to these branches.

(7.) Let the equation given be

$$y^2x + a^2x - a^3 = 0.$$

Whence

$$y = \pm\frac{a\sqrt{(a-x)}}{\sqrt{x}};$$

hence we have the conditions,

$$\text{when } x = a, \quad y = 0$$

$$\left.\begin{array}{l} +x > a, \\ \text{or } -x \end{array}\right\} y \text{ impossible}$$

$$+x < a, \quad \pm y \text{ real}$$

$$x = 0, \quad y = \infty.$$

The corresponding characters of the locus will be as in (Fig. 14.), taking $OB = a$.

The curve passes through B.

No part of the locus lies beyond.

The curve is symmetrical on each side of OB.

The axis Y becomes an asymptote.

(8.) To find a geometrical construction of the curve. From the equation we have

$$a^3x - a^2x^2 = y^2x^2,$$

$$\text{or } x\ (a - x) = \frac{y^2x^2}{a^2}.$$

But if a be the diameter of a circle, and x its abscissa, this is its equation, the ordinate being

$$\frac{yx}{a} = y_2, \text{ or we have } yx = y_2a;$$

that is, if a circle be described on OB as a diameter, and any ordinate $M\pi$ produced, so that the rectangle $OB \,.\, \pi M = OM \,.\, MP$; the locus of P will be the curve of the third order above investigated. This curve has been named the *witch*[a].

[a] The invention of an Italian lady, M. G. Agnesi.

(9.) Let the given equation be

$$y^2x=(a-x)\ (x-b)\ (x-c)\,;$$

$$\text{hence } y=\pm\sqrt{\left(\frac{(a-x)\ (x-b)\ (x-c)}{x}\right)}.$$

Case 1. Let the values of the constants be unequal, and $c>b>a$.

When $y=0$, we have $(a-x)\ (x-b)\ (x-c)=0$;

$$\text{or } \left.\begin{matrix}+y\\-y\end{matrix}\right\}=0 \text{ if either} \left\{\begin{matrix}x=a\\x=b\\x=c\end{matrix}\right.$$

If $x=0$, $y=\infty$.

If $b>a>x$, $y=\pm\sqrt{(+)\ (-)\ (-)}=\pm\sqrt{(+)}$ or is real.

$b>x>a$, $y=\pm\sqrt{(-)\ (-)\ (-)}=\pm\sqrt{(-)}$ ... impos.

$c>x>b$ $y=\pm\sqrt{(-)\ (+)\ (-)}=\pm\sqrt{(+)}$ ... real.

Hence (Fig. 15.) taking *Oa*, *Ob*, *Oc*, respectively equal to *a*, *b*, *c*, the locus meets the axis *X* at those points.

OY becomes an asymptote to the curve.

Between *O* and *a* the locus is symmetrical on each side of the axis.

Between *a* and *b* there is no locus.

Between *b* and *c* the locus reappears and forms a symmetrical curve returning into itself, or of an oval form.

At each of the intersections *a*, *b*, *c*, the ordinate becomes a tangent, since two values of *y* become $=0$ at the same time, and beyond that point become impossible.

But in this, as in most of the other cases, no exact determination of the form of the curve can be given without the help of the fluxional calculus.

(10.) Case 2. Let $b=c$,
the equation becomes when $y=0$

$$(a-x)\,(x-c)^2=0;$$

$$\text{or}\left.\begin{matrix}+y\\-y\end{matrix}\right\}=0,\text{ if either}\left\{\begin{matrix}x=a\\x=b=c\end{matrix}\right.$$

here the conditions remain as in the last case, except that the condition corresponding to the supposition $c>x>b$ vanishes. The nature of the locus consequently remains the same, except that the part between b and c vanishes, those points merging in one. (Fig. 16.) At this point two values of x becoming equal, and the adjacent values giving y impossible, it is called *a conjugate point*: it belongs to the locus algebraically, though it does not appear geometrically.

(11.) Case 3. Let $a=b$. The equation becomes when $y=0$, $(a-x)\,(x-a)\,(x-c)=0$,

$$\text{or}\left.\begin{matrix}+y\\-y\end{matrix}\right\}=0\text{ if either}\left\{\begin{matrix}x=b=a\\x=c\end{matrix}\right.$$

Here the condition corresponding to the supposition $b>x>a$ vanishes.

The other conditions remain. The form of the locus (Fig. 17.) differs from that in the last case. At the point where the two values of $x=a=b$ coincide, the branches of the curve intersect, and it is called a multiple point. The values of y do not here become impossible: the locus is continuous, and cuts the axis again in c; the part between a and c is called a *node*.

(12.) Case 4. Let $a=b=c$, the equation becomes when $y=0$ $(a-x)(x-a)^2=(a-x)^3=0$,

$$\text{and } \left.\begin{matrix}+y\\-y\end{matrix}\right\}=0, \text{ if } x=a=b=c.$$

$$\begin{array}{lll}\text{If } a>x, & y=\pm\sqrt{(+)} & \text{real.}\\ x>a, & y=\pm\sqrt{(-)} & \text{impossible.}\end{array}$$

Hence no part of the curve lies beyond the point corresponding to the coincidence of a, b, c, the node vanishes, and there is no conjugate point; (Fig. 18.) and since when $y=0$, two points of intersection with the axis merge in one, the axis becomes a tangent to the curve: and the same being the case for the branch on the lower side, the two branches touch, and their extremity is called *a cusp*.

The equation in this case being

$$y^2x=(a-x)^3,$$

if the origin be transferred to a, becomes

$$y^2(a-x)=x^3;$$

a form in which it is more commonly given.

(13.) In this case we can deduce a geometrical construction of the curve; for P being any point in the curve, $PM=y$, $AM=x$; and the equation gives

$$PM^2 . MO=AM^3.$$

$$\therefore \frac{AM^2}{PM^2}=\frac{MO}{AM}.$$

Describe a semicircle on AO, draw AP, let it meet the circumference in R, join OR, and we have a right angled triangle;

$$\text{Whence } \frac{AM^2}{PM^2} (\text{sim.} \triangle) = \frac{AN^2}{RN^2} = (\text{Euc. VI. 8.}) \frac{AN}{ON}.$$

Comparing this ratio with the former = to it, alternately we have

$$\frac{AN=AO-ON}{MO=AO-AM}=\frac{ON}{AM}\therefore=1$$

or if in the diameter of a circle we take AM always $=ON$, the intersection of AR with πM gives P in the curve. This curve is called the *cissoid of Diocles.*

COR. The same construction remaining we have by sim. Δs.

$$AM.NO=PM.RN \quad \therefore AM^2=\pi MP$$

(14.) From this property we may derive a simple mechanical construction of the curve, originally given by Newton. (Arith. Univ. Append. Sect. 46.)

(Fig. 19.) Assume two lines at right angles DC, CE, take two other lines also at right angles, the one $EB=DC$, the other BK indefinite, and let them be moved so that E is always in CE, and that BK always passes through D. The point P where BE is bisected traces out the cissoid.

With radius $CA=EP=AD$, describe a circle; through P draw the perpendiculars $\pi M\rho$ and PR; join $C\rho$: produce DB to meet the diameter produced in N.

From the right angle at B, and this construction, we have similar Δs EPR EBN. Whence,

$$\frac{PE=AC}{EN}=\frac{ER=\rho M}{BE=AO}=\frac{PR=CM}{NB=CN}$$

$$\text{by proportionals}=\frac{CA-CM=AM}{EN-CN=CE=\rho P}.$$

$$\text{Whence} \qquad P\rho M=OAM$$

$$\text{and subtracting equals} \quad \frac{-\rho M^2}{\pi MP}=\frac{-\rho M^2}{AM^2}$$

Which by the above Cor. is the property of the cissoid; consequently any point P constructed as above is a point in the curve.

The line CE is called the directrix.

(15.) The cissoid was invented with a view to the solution of the celebrated geometrical problem, the insertion of two mean proportionals between given extremes, on which the problem of the duplication of the cube depended.

Its application to the former of these problems readily follows from its geometrical construction.

We must first observe, that if with any quantities we have,

$$\frac{a^2}{b^2} = \frac{b}{d}$$

then b is the first of two means in continued proportion between a and d. For take c a third proportional to a and b: thence

$$\frac{a^2}{b^2} = \frac{a}{c} \therefore = \frac{b}{d}$$

$$\text{or} \quad \frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

Now (Fig. 18.) let AC CG be the given extremes, place them at right angles, and with centre C and radius CA describe a semicircle, and construct the cissoid.

Join OG and let it meet the cissoid in P, and let AP meet CG produced in Q. CQ is the first mean.

For from the curve $AM = NO$ and $MC = CN$, and by similar triangles,

$$\frac{AN}{AM} = \frac{RN}{PM} = \frac{PM}{NS} = \frac{\frac{1}{2}(RN + PM) = CQ}{\frac{1}{2}(PM + NS) = CG}$$

$$\text{also} \quad \frac{AN}{AM = NO} = \frac{AN^2}{RN^2} = \frac{AC^2}{CQ^2}$$

Whence, by the above lemma, CQ is the first mean required, and drawing the parallel GL, CL is obviously the second.

(16.) To find the polar equation to the cissoid:

The equation of coordinates may be put under this form

$$ay^2 - x(x^2 + y^2) = 0$$

or $ay^2 = x\sqrt{(x^2 + y^2)}.\sqrt{(x^2 + y^2)}$

$$\therefore \frac{ay^2}{x\sqrt{(x^2 + y^2)}} = \sqrt{(x^2 + y^2)}$$

but $\sqrt{(x^2 + y^2)} = r$, A being the pole:

also $\frac{y}{x} = \tan.\theta$ $\frac{y}{r} = \frac{y}{\sqrt{(x^2 + y^2)}} = \sin.\theta$

Whence by substitution,

$$r = a.\tan.\theta \sin.\theta$$

Which is the equation required; A being the pole, and AO the axis.

For further properties of this curve, and others related to it, the student is referred to Peacock's Examples on Diff. Calc. p. 166.

SECTION IV.

EQUATIONS OF THE FOURTH DEGREE.

(1.) General remarks, similar to those made at the beginning of the investigation of equations of the third degree, apply in this instance also. No writer has attempted the labour of classifying this order of curves; but it has been calculated that there are more than five thousand species. We shall here, as before, investigate only a few of the most important instances.

(2.) Let the equation be $yx^3 = a^4$

$$\text{or } y = \frac{a^4}{x^3}$$

Hence, since a^4 is always positive,

$$+ x \text{ gives } + x^3 \therefore + y$$
$$- x \ldots - x^3 \therefore - y$$

If $\pm x = 0 \quad y = \infty$ and if $y = 0 \quad x = \infty$.

Hence the locus lies on opposite sides of both the axes X and Y, which are also asymptotes. (Fig. 20.)

(3.) Let the equation proposed be,

$$x^4 - a^2x^2 - b^2x^2 + a^2b^2 - c^3y = 0$$

$$\text{whence } y = \frac{x^4 - a^2x^2 - b^2x^2 + a^2b^2}{c^3}$$

$$\text{or } y = \frac{(x^2 - a^2)(x^2 - b^2)}{c^3}.$$

Here (supposing $b > a$) we have

$$y = 0, \text{ if either} \begin{cases} x = -a \\ \quad = +a \\ \quad = +b \\ \quad = -b \end{cases}$$

If $x < a$	$y = (-)\,(-) = +$	
$a < x < b$	$y = (+)\,(-) = -$	
$b < x$	$y = (+)\,(+) = +$	
$x = \infty$	y	$+$

Hence the form of the locus is as in (Fig. 21.) taking from the origin parts $= +a, + b, - a, - b$; the curve lies above between a and a_2, below between $a\,b$ and a_2b_2, and above ad infinitum beyond bb_2.

(4.) If both a and $b = 0$, the equation gives

$$y = \frac{x^4}{c^3}$$

Hence the four points aa_2 bb_2 merge in one at O, which under these conditions is called a point of undulation.

The axis X is a tangent at O, and the curve extends above it in two infinite branches.

The curve in this case belongs to the general class of parabolæ.

(5.) Let the equation given be,

$$y^2x^2 + (x^2 - m^2)(x + b)^2 = 0$$

whence $y = \pm\sqrt{\left(\frac{(m^2 - x^2)(x + b)^2}{x^2}\right)}$

$$b \text{ being either} \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} m$$

If $x = \pm m$, then	$y = \pm 0$
$x > \pm m$	$y = \sqrt{(-)}$ impossible
$x < \pm m$	$y = \sqrt{(+)}$ real
$x = - b$	$y = 0$
$x = 0$	$y = \infty$
$b = 0$	$y = \pm\sqrt{(m^2 - x^2)}$
$m = 0$	$y = \pm\sqrt{(x + b)^2} = \pm x + b$

Hence to trace the locus: (Fig. 22.) From the origin O take $OD = +m$, $OD_2 = -m$, $OB = -b$. There are two loci corresponding to $+m, -m$. No part of the locus lies beyond D, D_2.

At these points two values of y vanish, and the ordinates become tangents.

Between O and D or D_2, the curve is symmetrical on each side of X.

If $b < m$, the locus on the negative side cuts the axis at B, and between B and D_2 forms a node.

The axis Y is an asymptote to both curves.

(Fig. 23.) If $b=m$, B and D_2 coincide, the node vanishes; and a cusp is formed: the other curve remaining the same.

(Fig. 24.) If $b > m$, the vertex D_2 lies between O and B; but b having still a real value, the condition $x=-b$ gives $y=0$; but $x=-m$ also continues to give $y=0$, and the intervening values are impossible; therefore B is a conjugate point.

If $b = 0$, B coincides with O, and the equation for the locus on the negative side becomes that of a circle on the diameter m.

When $m = 0$, D coincides with O: and the equation deduced is that of a straight line.

(6.) From this equation of coordinates to deduce the polar equation.

From the equation we obtain

$$x^2 (y^2 + (x+b)^2) - m^2 (x^2 + 2xb + b^2) = 0.$$

(Fig. 24.) Taking B as the pole, BP_2P meeting the two curves; $OM=PN=x \quad PM=y$:

$$r^2 = (x+b)^2 + y^2$$

$$r \cos. \theta = x + b \;\therefore\; x = r\cos.\theta - b.$$

Hence by substitution the equation becomes,

$$(r \cos. \theta - b)^2 r^2 - m^2 r^2 \cos.^2 \theta = 0,$$

or dividing by r^2, and extracting the root,

$$r \cdot \cos.\theta - b \pm m \cos.\theta = 0$$
$$\text{or } (r \pm m) \cos.\theta = b.$$

(7.) Hence we derive the geometrical construction: If the radius vector meet the axis in C, we have

$$BC \cdot \cos.\theta = b$$
$$\therefore BC = r \pm m$$

Hence if in the line BP, revolving about the fixed point B, the part CP or CP_2 be always of a constant magnitude, measured from the point C, at which the revolving line cuts the fixed axis Y, the point P or P_2 will trace out the locus of the equation just investigated.

This curve is called *the conchoid* of Nicomedes; the locus on the positive side being termed *the superior*, and that on the negative side *the inferior conchoid*. The axis Y is called the rule, and the constant part CP the modulus.

It is obvious that a mechanical construction may be made upon this principle: rulers BO OY being fixed at right angles; BP always passing through the fixed point B, and from the fixed point C in it, always moving in OY, CP CP_2 taken on each side at equal distances.

(8.) By means of the conchoid we can solve the problem, *to trisect a circular arc geometrically*.

Fig. (25.) Let AB be the arc to be trisected. Take the point B as the pole, and the diameter AD as the rule of an inferior conchoid, which passes through B, and cuts the circle again in P. Through P draw $P\pi$ parallel to AD: join $D\pi$, DB. The radius vector BPC gives the modulus $CP = DB = D\pi$: and this last, from the parallels, is parallel to CP. Hence $\angle$ $AD\pi = \pi PB =$ (by $\odot$) $\frac{1}{2}$ πDB: or $A\pi$ is one third of the arc AB.

The conchoid may also be used for finding two mean proportionals between given extremes. See Robertson's Conic Sections. Append. ed. 1802.

(9.) Let the equation given be,

$$(x^2+y^2)^2-a^2(x^2-y^2)=0$$

When $y=0$ we have $x^4-a^2x^2=0$,

which will be the case if either $\begin{cases} x=\pm a \\ x=\pm 0. \end{cases}$

Hence the locus cuts the axis twice at the origin, at a distance from it OV (Fig. 26.) $=a$, and again at an equal distance OV_2 on the other side $=-a$.

This curve is called the *lemniscata*[f]. But to trace the course of the locus would require the solution of the equation for y. It is more readily shewn from the polar equation, which is thus deduced:

(10.) The former equation may be put under this form:

$$x^2+y^2-\frac{a^2(x^2-y^2)}{x^2+y^2}=0.$$

Taking O for the pole we have

$$r^2=x^2+y^2 \quad y=r\sin.\theta \quad x=r\cos.\theta.$$

$$\therefore (x^2-y^2)=r^2(\cos.^2\theta-\sin.^2\theta)=(\text{Trigon.})\ r^2\cos.2\theta$$

Hence the equation becomes

$$r^2-a^2\cos.2\theta=0$$

which is the polar equation required.

Hence when $r=0$, $\cos.2\theta=0$

$$\therefore 2\theta_0=\tfrac{1}{2}\pi, \text{ or } \tfrac{3}{2}\pi, \text{ or } \tfrac{5}{2}\pi;\ \&c.$$

that is, the locus lies in alternate quadrants, and is included between two lines at right angles, and forming

[f] Invented by James Bernoulli.

half right angles with the axis at the origin, where they are also tangents to the curve.

(11.) Hence we derive the geometrical construction.

$$\text{Let } d=\frac{a^2}{r}, \text{ or } rd=a^2$$

whence, from the polar equation, by transposition,

$$r=d.\cos.2\theta.$$

Now if we take an equilateral hyperbola, having its semi-axis$=a$, we have by Conic Sections (I. ii. 36.)

$$CR.CP=a^2,$$

$$\text{or writing } CP=d\ CR=r, \ldots rd=a^2.$$

Also in the same curve by [C. S. I. ii. 37.] writing the $\angle PCM=\theta$ we have $\angle PCR=2\theta$: and consequently from the right angled triangle, $r=d.\cos.2\theta$.

$$\therefore r^2=a^2\cos.2\theta.$$

Thus the locus of the *concourse of the central perpendicular, and the tangent to the equilateral hyperbola,* is determined by the same polar equation as the lemniscata, with which it is consequently identical.

This curve is, however, only one species of an extensive class; for an account of which the student is referred to Lardner's Alg. Geom. p. 345, &c. and to Peacock's Examples on Diff. Calc. p. 168, &c.

The fifth and higher degrees of equations present no examples of importance: we shall therefore proceed to the general properties of curves of the nth degree.

SECTION V.

EQUATIONS OF THE *N*TH DEGREE.

(1.) A general and complete equation of the *n*th degree, is one consisting of terms which involve the variables x and y in all the combinations of the powers of each, such that the highest sum of the exponents $=n$, each term having a constant coefficient. These combinations are easily exhibited; and most clearly so, when arranged in the form subjoined, called the analytical triangle[g]:

y^n	$y^{n-1}x$	$y^{n-2}x^2$	$y^{n-3}x^3$	—	—	yx^{n-1}	x^n
y^{n-1}	$y^{n-2}x$	$y^{n-3}x^2$	—	—	yx^{n-2}	x^{n-1}	
—	—	—	—	—	—		
—	—	—	—	—			
y^3	y^2x	yx^2	x^3				
y^2	yx	x^2					
y	x						
1							

[g] De Gua's improvement on a similar arrangement by Newton.

The whole number of terms are those of the complete equation of the nth degree; the whole minus the upper row, those of the $(n-1)^{\text{th}}$ degree, &c. The four lowest give the third degree; the three lowest the second; the two lowest, the first: or in general, *reckoning from the bottom, the first* $(n+1)$ *rows give all the terms of a complete equation of the* n*th degree.*

(2.) Hence we readily find *the number of terms* in a complete general equation of the nth degree.

For the number of terms in each row of the table, reckoning from the bottom, increases by 1; each *row* therefore is a *term* in an arithmetical series, whose first term is 1, and common difference 1, and the number of rows, corresponding to the equation of the nth degree, is $n+1$. Hence the *number of terms in that equation* is found by summing the series; or we have (Wood, Alg. 212.)

$$s=\frac{(n+1)}{2}.(2+n)$$

(3.) The number of constant quantities usually given as coefficients is the same as the number of terms: but this number may be diminished by one without affecting the conditions of the equation; since all the terms may be divided by the coefficient of any one term, and the whole remain $=0$. Thus *the whole number of determinate, independent coefficients, is one less than the number of terms.* In an equation of the nth degree we consequently have the number of such coefficients=

$$\frac{(n+1)\ (n+2)}{2}-1\ =\frac{n\,(n+3)}{2}.$$

If two equations of the same degree have their corresponding coefficients *proportionals,* they are *identical:* since dividing all the terms of each by the coefficient of the first term, the corresponding coefficients will be equal fractions.

(4.) Def. An equation is said to be *homogeneous* when the sum of the exponents in each of its terms is the same.

All equations representing loci must be homogeneous.

For in such equations the object is to express the value of one variable in terms of the other and the constants; each variable representing linear extension. And to construct the locus, we solve the equation for one of the variables, as for y, and obtain an expression of the form

$$y = \sqrt[n]{f(x)}:$$

but since y represents linear extension it is of *one dimension;* therefore its value is so likewise: or the expression $\sqrt[n]{f(x)}$, whatever be its form, must have all its terms of one dimension only: $\therefore f(x)$ is homogeneous also, and each term of n dimensions.

The constant coefficients in an equation, though commonly expressed by simple quantities, are always understood to represent quantities of such dimensions as shall render the term homogeneous with the others.

(5.) To determine the number of points through which an algebraic curve of the nth degree may be drawn.

Suppose the coordinates of an indefinite number of points given, as $x_2 y_2$, $x_3 y_3$, &c. and the equation to the curve to have its coefficients undetermined: then, since the curve is by supposition to pass through the given points, the coordinates of those points will be identical with those of the curve. If, then, we substitute successively the values of these coordinates in the general equation to the curve, we shall have,

$$Ay_2^n + By_2^{n-1}x_2 + Cy_2^{n-2}x_2^2 + \quad = 0$$
$$Ay_3^n + By_3^{n-1}x_3 + Cy_3^{n-2}x_3^2 + \quad = 0$$
&c. &c.

and there will obviously be as many such equations as we assume points. But the whole number of coefficients was before shewn to be $\frac{n(n+3)}{2}$. Therefore to determine these there must be the same number of such equations as those above, in which $x_2 y_2$, &c. are supposed given; and consequently the same number of points assumed: that is, a curve of the nth degree may be drawn to pass through $\frac{n(n+3)}{2}$ given points.

Since each of these equations for the indeterminate quantities A, B, &c. are simple equations, the values can never be impossible.

An unlimited number of curves of the nth degree may be drawn, fulfilling the condition of passing through a less number of points than $\frac{n(n+3)}{2}$. Since only as many of the coefficients will be determined as there are points.

(6.) To find the greatest number of points in which a straight line can meet an algebraic curve.

It has already appeared that changes in the axes affect only the *form* of the equation, and not its *degree*, nor consequently the degree or nature of the curve. The axes may therefore always be supposed so assumed, that the form of the equation of a given degree may be complete, with all the terms. The general form of such an equation is readily seen by referring to the table before given, and supposing the coefficients supplied. If in that form we now suppose y to become $= 0$, the equation will be reduced to the terms involving only the powers of x, or will be of the form,

$$Mx^n + Nx^{n-1} + Px^{n-1} \ldots\ldots + Vx + W = 0.$$

This equation being solved, each real root of it gives a value of x corresponding to $y = 0$, or a point where the locus intersects the axis X. The number of real roots, and consequently of such points, cannot exceed n. (Wood's Alg. Part II.) As some of the roots may always be impossible, and, when n is even, all of them may be so, in these cases there will be fewer points of intersection than n; or none at all. Thus the *greatest* number of points in which a curve of the nth degree can intersect its *axis, is n.*

The assumption here made respecting the axes is that which gives x with the *highest* exponent which the equation admits in the form which it takes when $y = 0$. No other axes therefore can give *more* points of intersection; or, since the number of positions which the axes may assume is unlimited, *no straight line can have more than* n *points of intersection, with a curve of the* n*th degree.*

(7.) If the dimension of a curve be an odd number, it must have at least two infinite arcs. For equations of odd dimensions have always at least one real root, and (Wood's Alg. Part II.) consequently for every value of one of the coordinates the equation must at least give one real value of the other.

If the nature of the equation be such as not to assign any limit to the values of x, then if any value whatever be given to $+x$ or $-x$, that is, in the construction, if x be increased ad infinitum on either side of the origin, there will be a corresponding real and infinite value of y, or the locus will extend ad infinitum on each side of the origin.

If the nature of the equation be such that x cannot exceed a certain constant value without rendering y impossible, since the general property will hold good respecting one of the coordinates, we shall have for every value assignable to y ad infinitum, both ways, a real value of x.

Of the former case we have had examples in the first degree; in the cubical parabola, &c.; of the latter, in the witch, the cissoid, &c.

If the dimension be an even number, as the roots of the equation may all become impossible, certain values of both coordinates may give no real values of the other, or the figure of the locus may be entirely limited between certain tangents. We have instances of this in the ellipse and in the lemniscata.

(8.) Curves whose equations are of the form

$$y^n = px^m$$

are termed parabolæ of the *n*th order.

In this class of curves, (p being supposed always positive,) if m be an *even* number,

the condition $\pm x$ gives $+px^m$ and $\therefore$ $+y$
and . . . . . $\pm x = \infty$ . . . . $+y = \infty$

or there are two infinite arcs on the *same* side of X.

If m be an *odd* number,

$\pm x$ . . . . $\pm y$
and $\pm x = \infty$ . . . . $\pm y = \infty$

or there are two infinite arcs on *opposite* sides of X.

If the equation were of the form

$$y^n = px^m + qx^{m-1} + \text{ \&c.}$$

$$x = \infty \text{ would give } px^m > qx^{m-1} + rx^{m-2} + \text{ \&c.}$$

and the sign of $y = \infty$ would still depend upon that of px^m.

(9.) Curves whose equations are of the form

$$y^n x^m = a^{n+m}$$

are termed hyperbolæ of the $(n+m)$th order.

In all these curves the axes form asymptotes, since when

$$x = 0 \quad y = \infty$$
$$\text{and } y = 0 \quad x = \infty$$

From the equation we have

$$y^n = \frac{a^{n+m}}{x^m}$$

Hence if m be an *even* number, and a be supposed always positive,

$$\pm x \text{ gives } + x^m \text{ and } \therefore + \frac{a^{n+m}}{x^m} \therefore + y^n$$

and $\pm x = 0$ or ∞ . . . . $+y = \infty$ or 0,

or there are two infinite arcs on the *same* side of the axis X.

If m be an *odd* number,

$\pm x$ gives $\pm x^m$ . . . . $\pm y^n$

and $\pm x = 0$ or ∞ . . . . $\pm y = \infty$ or 0,

or the two infinite arcs lie on *opposite* sides of X.

These two extensive classes of curves are both included under the still more general formula

$$y^n = px^{\pm m}$$

which with $-m$ becomes $\Big\}\ y^n x^m = p.$

(10.) To find the greatest number of points in which a curve of the *n*th degree can intersect one of the *m*th degree.

If either of the axes be parallel to a line joining any two points of intersection, we should have two values of x, or of y, equal. Let the axes therefore be so assumed as not to be parallel to any such line of junction, and thus to give distinct values of the coordinates X and Y, for every point at which they are common to the two curves, that is, for every point of intersection.

Let the equations be such as to give for any one such value of x, and the corresponding value of y,

$$y = Fx^n + Gx^{n-1} + \ldots\ldots + Kx$$
$$x = Sy^m + Ty^{m-1} + \ldots\ldots + Zy$$

Hence by substitution,

$$x = S\,(Fx^n + \ldots + Kx)^m + \&c.$$

an equation which is necessarily of *mn*th degree; and consequently the greatest number of real roots it admits is *mn;* which will therefore be the greatest number of intersections.

(11.) Let a curve of the *n*th degree intersect its axes; the successive points at which these intersections take place upon the axis x are determined by the values of x, which give $y = 0$, and similarly upon the axis y by the values of y, which correspond to $x = 0$.

When $y = 0$, as in a former case, the general equation is reduced to

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Vx + Z = 0;$$

each real root of which, as before observed, gives a value of x corresponding to $y = 0$, or a point of intersection. And in any such equation, dividing by A, and transposing, we have,

$$x^n + \frac{B}{A}x^{n-1} + \frac{C}{A}x^{n-2} + \ldots\ldots + \frac{V}{A}x = \frac{-Z}{A}$$

But from the nature of equations, (Wood's Alg. Part II.) this expression results from the successive multiplication together of all the roots, and the value of that product $= \frac{-Z}{A}$.

In like manner taking the equation for y when $x = 0$, we should have,

$$My^n + Ny^{n-1} + Py^{n-2} + \ldots. + Wy + Z = 0,$$

$$\text{or} \quad y^n + \frac{N}{M}y^{n-1} + \frac{P}{M}y^{n-2} + \ldots. + \frac{W}{M}y = \frac{-Z}{M}.$$

Consequently the ratio of the continued product of the segments of x, reckoned from the origin, to that of the segments of $y = \frac{ZM}{AZ} = \frac{M}{A}$, *or these products are inversely in the constant ratio of the coefficients of the highest powers of x and y* in the above equations.

(12.) If we now suppose the axes with the same angle of ordination transferred to a *new origin*, and continuing parallel to themselves, it has already appeared that this does not affect the *degree* of the equation, nor the value of the coefficicnts of x^n and y^n; hence the ratio $\frac{M}{A}$ remains unaltered: and the same will hold good for any successive new origins.

These changes in the supposition are the same as if, instead of altering the position of the axes, we had taken successive systems of secants parallel to the original axes. And we have thus the general result, that *if we take any two secants meeting each other and pa-*

rallel to the axes, the ratio of the continued products of their segments reckoned from their concourse; is equal to that of the corresponding products of the segments of any two parallel secants.

If two or more roots are equal, the corresponding points of intersection merge in one, and we have the square, cube, &c. of the tangent, instead of the product of the segments of the secants.

For instances of the application of this very important theorem the reader is referred to Lardner's Algebraic Geometry, Arts. 138. 606. And for its accordance in curves of the second degree with investigations purely geometrical deduced from totally different principles, to Dr. Robertson's Conic Sections, Book I. compared with Lardner, p. 66, &c.

(13.) If the general equation be arranged by dimensions of y, it will be of this form; (dividing by A)

$$y^n + \frac{Bx+C}{A} y^{n-1} + \left(\frac{Dx^2+Ex+F}{A}\right) y^{n-2} + \ldots + \frac{N}{A} = 0$$

And as for any assumed value of x, this expression results from the continued multiplication of the several values of y, (Wood's Alg. II. 271.) from the nature of such multiplication it is evident that the coefficient of the second term = the sum of all such values. Also the number of such values is (n), and consequently the *mean* value will be

$$\frac{1}{n} \cdot \frac{Bx+C}{A}.$$

If the sum of all the positive values of y equal the sum of all the negative, the mean values must also destroy each other; or,

$$\frac{Bx}{nA} = \frac{C}{nA}.$$

Hence if a line, taken as the axis X, be so drawn as to divide the ordinate y in such a manner that for all values of x the mean values of y on each side of it should be equal, the equation of this line must give

$$y = \frac{-(Bx + C)}{nA},$$

or its equation will be $nAy + Bx + C = 0$. Such a line is called a *diameter*, which is thus more generally defined to be *a line intersecting a system of parallel chords, so that the sums of the segments between it and the several points where the same ordinate meets the curve on each side are equal.*

(14.) The coefficient of the third term in the above form gives for the same value of x the *sum of the products of every two values* of y; and the number of such products will be $\frac{n(n-1)}{2}$.

Hence, upon the same principles, a line of the second degree whose equation is

$$\frac{n(n-1)}{2} Ay^2 + (n-1)\ (Bx + c)y + Dx^2 + Ex + F = 0$$

will divide the ordinates, so that the sum of the positive products of the segments will equal the sum of the negative.

This curve will also have the same diameter as the curve of the nth degree.

In like manner curves of each successive degree may be shewn to have similar properties regarding the products of every three values of y, &c. Such curves are termed *curvilinear diameters.*

For further information on these points the student is referred to Lardner's Algebraic Geometry, Sect. XXI.

(15.) In the curves before discussed we have had several instances of multiple and conjugate points, &c.; we have also found a point of contact to result from two points of intersection merging in one, owing to two values of y becoming equal, and beyond that point being impossible. These conditions may be taken in a more general point of view from the consideration that since two roots of an equation always become impossible together, (Wood's Alg. p. 277.) in the equation which determines y, at any value of x where y becomes impossible, *an even number* of values of y vanish together, and the corresponding points at which the ordinate cuts the curve form a point of contact. If two values vanish, it is a simple point of contact; if four, six, &c. it is called a point of simple, double, &c. *undulation.* If an *odd* number become equal, some value of y continues real, but with a different sign: if three, five, &c. it is called a point of simple, triple, &c. *inflexion.*

But the conditions of all these and similar points in a curve must be determined by the fluxional calculus.

(16) An equation of the nth degree may be of such a form as to be capable of being resolved into two or more rational factors, which are equations of inferior degrees, and the sum of whose dimensions $= n$.

Of this an example occurred in the general discussion of the second degree. (Sect. ii. 6. Case 5.)

In these cases the locus is *not one curve* of the nth degree, but *several of inferior orders;* such, and so many, that the sum of their degrees $= n$.

In any such case if the factors be $=$, the equation resulting being a complete power, will only represent one line whose equation will be the root of that power. These cases are therefore understood to be excepted, when we speak generally of the locus of an equation of the *n*th degree.

(17.) Since an equation of the *n*th degree may be so derived as to represent a number of separate loci of inferior degrees, let such an equation be that of *n right lines,* and suppose it put into the following form:

$$y^n + (\alpha x + \beta)\, y^{n-1} + (\gamma x^2 + \delta x + \epsilon)\, y^{n-2} + \&c. = 0 \ldots (M).$$

Then for any value of x the sum of the values of y is $\alpha x + \beta$; for since the equation results from the multiplication together of the simple equations belonging to the straight lines which are of this form,

$$\begin{aligned} &y + ax + b = 0 \\ &y + cx + d = 0 \\ &\&c. \quad \&c. \end{aligned}$$

whence we have by multiplication,

$$y^n + \{(a+c+e+\ldots.)\,x + (b+d+f+\ldots.)\}y^{n-1} + \ldots. = 0$$

it follows that these coefficients are the same as those at first assumed; and since the sum of the values of y in the above equations is $(a+c+e\ldots)\,x+(b+d+f\ldots)$, we have its equal $\alpha x+\beta =$ this sum.

(18.) If we have a *curve of the* n*th degree* represented by the equation, put in a similar form,

$$y^n + (Ax + B)\, y^{n-1} + (Cx^2 Dx + E)\, y^{n-2} + \ldots = 0 \ \ldots (N)$$

we have here, as before, $Ax + B =$ the sum of the values of y corresponding to any value of x.

(19.) If the curve have as many asymptotes as it has dimensions, we shall have the values of y in the curve when $x = \infty$ coinciding in the limit with those in the equation of the asymptotes: and for these points the equations coincide. Hence if the former equation be made up of the equations of n right lines which are asymptotes to the curve, when $x=\infty$,

The equation (M) becomes

$$y^n + axy^{n-1} + \ldots = 0.$$

The equation (N) becomes

$$y^n + Axy^{n-1} + \ldots = 0.$$

and these being identical $a=A$.

Also in all cases $ax+\beta$, and $Ax+B$ are the sums of the values of y, belonging to the respective loci: hence by substitution these quantities become $Ax+\beta$, $Ax+B$, and their difference $\beta-B$ is invariable for all values of x.

But when $x=\infty$ this difference$=0$. Consequently at all other values it$=0$, or $\beta=B$; and therefore at all values of x we have

$$ax+\beta=Ax+B;$$

or, if an ordinate be drawn from a point in the axis corresponding to any given value of x, and meeting the several branches of the curve and the asymptotes, we have *the sum of the ordinates to the curve equal to the sum of the ordinates to the asymptotes*; or, for the sake of distinction, writing the values of the ordinates to the curve y, y_2, y_3, &c. and to the asymptotes z, z_2, z_3, &c.

$$y+y_2+y_3+y_4+\ldots = z+z_2+z_3+z_4+\ldots.$$

Hence by transposition

$$(y-z)+(y_3-z_2)+\ldots = (z_2-y_2)+(z_4-y_4)+\ldots.$$

or the sum of the parts of the ordinate intercepted between the first asymptote and first branch of the curve, and between the third asymptote and third curve, &c. is equal to the sum of the parts intercepted between the second asymptote and second curve, and between the fourth asymptote and fourth curve, &c.

An instance of this in lines of the second degree is found in the familiar property of the asymptotes of the hyperbola. Conic Sections. (II. i. 6.)

DIVISION II.

TRANSCENDENTAL EQUATIONS.

(1.) The distinction between algebraic and transcendental equations was before pointed out. (Int. 9.) The transcendental class includes all equations not reducible to a form involving only algebraic functions of the variables, or which when reduced into such a form, consist of a series having an infinite number of terms; as is obviously the case where such expressions as sines, tangents, logarithms, &c. are involved. The loci of these equations are sometimes called *mechanical* curves. For this class of curves there are no general principles of arrangement, or investigation, except so far as the transcendental functions are classed under the heads of trigonometrical, logarithmic, &c.; we shall consider only a few of the most important instances.

§. 1.

TRANSCENDENTAL EQUATIONS INVOLVING TRIGONOMETRICAL FUNCTIONS.

(2.) Let the given transcendental equation be

$$y - r \cos.\left(\frac{x - n\sqrt{(2\,ry - y^2)}}{mr}\right) - r = 0,$$

in which the quantity between the brackets represents the value of a circular arc dependent upon x and y, n and m being arbitrary quantities, and r a constant radius.

Writing this arc $= \psi$, we have

$$\psi = \frac{x}{mr} - \frac{n\sqrt{(2\,ry - y^2)}}{mr}$$

and $y = r + r\cos.\psi = r\,(1 + \cos.\psi)$ . . . . . (A)

$$\text{whence } \psi = \frac{x}{mr} - \frac{n\sqrt{(2\,r^2\,(1 + \cos.\psi) - r^2\,(1 + \cos.\psi)^2)}}{mr}$$

$$= \frac{x}{mr} - \frac{n\sqrt{(r^2\,(1 - \cos.^2\psi))}}{mr}$$

$$= \frac{x}{mr} - \frac{n\sqrt{(r^2\sin.^2\psi)}}{mr}$$

or $rm\,\psi = x - nr\sin.\psi$

$\therefore x = r\,(m\,\psi + n\sin.\psi)$ . . . . (B)

(3.) To investigate the locus. In the first place, it is evident, that for each value of x there is only one value of y. It also appears from the above expressions that if $y = 2\,r$, the equation becomes

$$r - r\cos.\frac{x}{mr} = 0$$

$$\therefore\ 1 = \cos.\frac{x}{mr}\ \therefore\ \frac{x}{mr} = 0\ \therefore\ x = 0.$$

(4.) Hence, with rectangular axes, taking a value of $y = 2r$, and upon it, as a diameter, describing a circle, we have at this point $x = 0$. And for any other value of y, as (Fig. 27.) $O\mu = PM$, we have the value of $OM = x$ dependent upon an arc ψ of the circle on OV, as $O\pi$, whose sine, to radius r, $= \mu\pi$. This sine being drawn and produced, the value of x or $\mu P = OM$ is found by taking the part produced according to the conditions of the expression above deduced for x: that is, according to the relative values of m and n. Among the various cases which might arise, the following are the only ones of importance:

If we have		The equation becomes
$n = 1$ and	(1) $m = 1$	$y - r\cos.\left(\frac{(x - \sqrt{(2ry - y^2)}}{r}\right) - r = 0$ and $y = r(1 + \cos.\psi)$; $x = r(\psi + \sin.\psi)$
	(2) $m > 1$ or (3) $m < 1$	$y - r\cos.\left(\frac{x - \sqrt{(2ry - y^2)}}{mr}\right) - r = 0$ and $y = r(1 + \cos.\psi)$; $x = r(m\psi + \sin.\psi)$

(4) $n = 0$ $m = 1$ $\left\{\begin{array}{l} y - r\cos.\left(\frac{x}{r}\right) - r = 0 \\ \text{and } y = r\ (1 + \cos.\psi)\ ;\ x = r\psi. \end{array}\right.$

In the first case $x = \mu\pi + \pi P$ $\mu\pi = r\sin.\psi$ $\therefore \pi P = r\psi$
second and third, ... $\pi P = rm\psi$
fourth, ... $\mu P = r\psi$

or in the first case P is determined by taking $\pi P=$ the length of the arc ψ: in the second and third, in the given ratio m to that length, greater or less according as $m >$ or $<$ than 1: in the fourth, $\mu P=$ the length.

But $\mu\pi=\sin.\ \psi$ being the sine both of $O\pi$ and $V\pi$, since when $y=2r$ $x=0$, and $\because$ at this point the arc $\psi=0$ also, it follows that the value of πP must be measured by the arc $V\pi$: hence the construction of the curve in the different cases.

In the first case, when πP is always $=$ the length of the arc $V\pi$, the curve is called *the common cycloid*[a]. In the second and third cases, it is termed the *prolate cycloid*, when m is a fraction >1; and the *curtate* when <1. In the fourth case, where the part equal to the sine of the arc vanishes, and the portion is measured off from $\mu=$ the length of the arc, the curve is called the *companion to the cycloid.*

(5.) If in any of these cases the origin be transferred from O to the point A, where the curve meets the axis X, (called the base,) the form which the equation will assume is readily found.

Corresponding to the point A, on the former supposition, it is evident that $\psi=\pi$: but this point now being taken as the origin, the value of ψ must be measured from the base instead of V, or we must substitute for ψ

$$\psi_2=\pi-\psi$$

Hence the former value of y must be changed into

$$y=r\ (1+\cos.\ (\pi-\psi),$$
$$\text{or } y=r\ (1-\cos.\ \psi_2)\ .\ .\ .\ .\ .\ .\ .\ (C).$$

[a] This curve was originally invented by Galileo: its properties, and those of other species, were investigated by Des Cartes, Pascal, Sir C. Wren, Wallis, and others.

The new value of x may be thus found: on the former supposition, when the curve meets the base, or when $y=0$, the equation becomes

$$-r \cos. \frac{x}{mr} - r = 0$$

$$\text{or } \cos. \frac{x}{mr} = -1 \quad \therefore \; x = \pi mr.$$

Writing the new value measured from $A = x_2$, we have

$$x_2 = \pi mr - x,$$

or by substitution $= \pi mr - r\,(m\psi + n \sin.\,\psi)$,

$$= r\,(m\,(\pi - \psi) - n \sin.\,\psi):$$

$$(\text{or since } \sin.\,\psi = \sin.\,\psi_2)$$

$$x_2 = r\,(m\psi_2 - n \sin.\,\psi_2) \; . \; . \; . \; . \; . \; (D)$$

$$\therefore \; \psi_2 = \frac{x_2 + nr \sin.\,\psi_2}{mr}$$

Whence (suppressing the distinctive marks) the equation becomes,

$$y + r \cos. \left(\frac{x + n\sqrt{(2ry - y^2)}}{mr}\right) - r = 0.$$

(6.) The condition which gives $y=0$, as above, is now $\cos. \frac{x}{mr} = +1 = \cos.\,2\pi.$ which also evidently belongs equally to every successive complete circumference. Hence the locus meets the base in an infinite number of points, corresponding to values of x, equal to successive circumferences of the circle.

(7.) The origin being at A, in the case $n=1$, $x = rm\psi - r \sin.\,\psi$: on the axis OV, with the same centre, conceive a circle described with radius $= rm$, and through the extremity R of its diameter draw a parallel to the base. *Let a part of this line A_2R be taken from A_2*, (corresponding to A,) $= x + r \sin.\,\psi \;\therefore = rm\psi$, or *equal the length of the arc* ψ of the circle whose radius is rm.

Let the circle (Fig. 27.) described on the axis OV be conceived to move parallel to itself till π coincides with P, the construction in other respects remaining unaltered. Let this condition be represented in the other figures, (28, 29, 30,) and since $A_2R = x + r \sin. \psi$, the perpendicular diameter of the circle comes into the position R; and this being the case for any point P in the curve, *A_2R always = the length of the arc, intercepted between R and the radius through P*; hence the result will be the same as if *every point in that arc had been successively in contact with every point in A_2R*. Or the same locus would be described if *the circle with radius* mr *were supposed to roll along A_2R*, and the extremity of the *radius* $= r$ to be the point tracing out the curve; which, according as the extremity of the radius r lies

$$\left.\begin{matrix}\text{upon}\\ \text{within}\\ \text{without}\end{matrix}\right\}\text{the circle } mr\text{, will be the }\left\{\begin{matrix}\text{common}\\ \text{prolate}\\ \text{curtate}\end{matrix}\right\}\text{cycloid.}$$

(8.) The mode of constructing the cycloids mechanically, just stated, leads to a *geometrical* extension of this class of curves, by supposing the base on which the circle rolls to be no longer a straight line, but itself a curve.

Thus if we take a circle as the base, and suppose another circle to roll upon its circumference, and, as before, take any point upon, within, or without its circumference, it will trace out a curve which is termed an *epitrochoid:* or if the point be *on the circumference*, an *epicycloid*[b].

Again, corresponding curves may be in like manner

[b] Invented by Roemer: investigated by Newton, John Bernoulli, &c.

traced out by supposing the second circle to roll on the inner or concave side of the circumference of the first, in which case the curve is termed an *hypotrochoid*, or, if the point be on the circumference, an *hypocycloid*.

(9.) To find the form of the equation in these cases:

(Fig. 31.) If we suppose the revolution to commence from A_2, upon the circumference of the circle A_2D at any point R; join the centres OC, and let $CP = r$, the point P tracing out the locus, and the radius $CR = rm = C\rho$ in CP produced. Then it is evident that the space traversed, or $A_2R = R\rho$.

Let the radius of the circle $A_2D = R$, then from the equality of the arcs

$$A_2R = R\phi = R\rho = rm\psi$$

$$\text{whence } \psi = \frac{R\phi}{rm}$$

Drawing the perpendicular $C\mu$, and PM CM_c perpendicular to OM,

$$\angle CP\mu = \psi + \phi = \frac{R\phi}{rm} + \phi$$

$$= \frac{R+r}{rm}\phi$$

$$C\mu = r \sin. CP\mu = r \sin. \frac{R+rm}{rm}\phi$$

$$P\mu \quad = \qquad = r \cos. \frac{R+rm}{rm}\phi$$

$$CM_c = (R+rm) \cos. \phi$$

$$OM_c = (R+rm) \sin. \phi$$

Hence we have,

$$y = (R+rm) \cos. \phi + r \cos. \frac{R+rm}{rm}\phi \quad . \quad . \quad . \quad (E)$$

$$x = (R+rm) \sin. \phi + r \sin. \frac{R+rm}{rm}\phi \quad . \quad . \quad . \quad (F)$$

In the case of the epicycloid $rm = r$, and these forms become

$$y = (R+r)\cos.\phi + r\cos.\frac{R+r}{r}\phi$$

$$x = (R+r)\sin.\phi + r\sin.\frac{R+r}{r}\phi$$

For the hypotrochoid and hypocycloid the equations are the same, except that r and $\therefore rm$ are now negative.

(10,) About the centre O describe a circle with a radius $OA = OC - r$, at which distance the locus will terminate; to find the angle A_2OA intercepted between the extremity of the curve and the position OV; that is, the value of ϕ corresponding to the position A or B.

For either of the points A, B, the values of x and y being taken, we have

$$x^2 + y^2 = OA^2 = (R + mr - r)^2$$

Substituting for x^2 and y^2 their values in the equations (E) and (F), this becomes,

$$\left\{\begin{aligned} &(R+rm)^2\cos.^2\phi + 2(R+rm)\cos.\phi . r\cos.\frac{R+rm}{rm}\phi + \\ &\quad r^2\cos.^2\frac{R+rm}{rm}\phi \\ &+ (R+rm)^2\sin.^2\phi + 2(R+rm)\sin.\phi . r\sin.\frac{R+rm}{rm}\phi + \\ &\quad r^2\sin.^2\frac{R+rm}{rm}\phi \end{aligned}\right.$$

$$= (R+rm)^2 - 2r(R+rm) + r^2$$

But the first member of this equation, by collecting terms and substituting $\cos.^2 + \sin.^2 = 1$, becomes

$$(R+rm)^2 + r^2 + 2r(R+rm)\left(\cos.\phi\cos.\frac{R+rm}{rm}\phi + \sin.\phi\sin.\frac{R+rm}{rm}\phi\right)$$

whence $-2r(R+rm)=2r(R+rm)\cos.\left(\frac{R+rm}{rm}\phi-\phi\right)$

$$\text{or } -1=\cos.\left(\frac{R}{rm}\phi\right) \therefore \frac{R}{rm}\phi=\pi, \text{ or } \phi=\frac{rm}{R}\pi.$$

(11.) If we have $rm=R$ $\phi=\pi$, or the extremities of the curve coincide. If we have also $rm=r$, or the curve be an epicycloid, and the radius of its generating circle = the radius of the base, the curve is called the *cardioid*. In this case the equations above become,

$$y=R\,(2\cos.\phi+\cos.2\phi)$$
$$x=R\,(2\sin.\phi+\sin.2\phi)$$

For several further properties and relations of these curves, see Peacock's Examples, p. 186, &c.

(12.) Let there be given the equation

$$y=(r-x)\tan.\frac{\pi x}{2r}$$

Here, if $x=o$, $y=o$.

Setting out from these values, we shall have the conditions corresponding to successive values of x, thus:

Values of x.

$0<x<r$ — As x increases y increases.
Also we have $+(r-x)$
and $\frac{\pi x}{2r}<\frac{\pi}{2}$ $\therefore$ $+\tan.$ $\Big\}\therefore +y$

$x=r$ — $y=0\,.\tan.\frac{\pi}{2}=0\,\infty$ . . . $=a$
For the actual value of which, recourse must be had to the calculus.

$r<x<2r$ — . . . . y decreases.
$-(r-x)$
$\frac{\pi x}{2r}>\frac{\pi}{2}$ $\therefore$ $-\tan.$ $\Big\}\therefore +y$

$x = 2r$ . . . . $y = 0$

$2r < x < 3r \left\{ \begin{array}{l} \text{. . . . } y \text{ increases.} \\ \left. \begin{array}{r} -(r-x) \\ \frac{\pi x}{2r} > \pi \therefore \; + \tan. \end{array} \right\} \therefore -y \end{array} \right.$

$x = 3r \left\{ \; y = -2r \tan. \frac{\pi 3r}{2r} \right.$ . . . $= \infty$

$3r < x < 4r \left\{ \begin{array}{l} \text{. . . . } y \text{ decreases.} \\ \left. \begin{array}{r} -(r-x) \\ \frac{\pi x}{2r} > \frac{3\pi}{2} \therefore \; -\tan. \end{array} \right\} \therefore +y \end{array} \right.$

$x = 4r \left\{ \; y = -3r \tan. \frac{\pi 4r}{2r} \right.$ . . . $= 0$

&c. &c.

This series of values might evidently be continued ad infinitum; y always changing its sign when x becomes = successive multiples of r, becoming = 0 at even, and = ∞, at odd, multiples. The same would be the case with a series of values of $-x$.

(13.) Hence we readily deduce the construction of the curve. (Fig. 32.)

Taking rectangular coordinates, let A be assumed as the origin. Take a value of x, $AC = r$, and assuming any other value as AM, draw a perpendicular MP, on which the corresponding value of y may be found by taking PM such that to radius $(r-x)$ or CM, we may have,

$$y = \tan. \frac{\pi x}{2r} = \tan. \psi$$

or so that a radius CP revolving about C shall form with the axis AX an angle ψ, determined by the proportion

$$\frac{x}{r} = \frac{\psi}{\frac{1}{2}\pi}.$$

Thus the point P, determined by PM moving parallel to itself, and intersecting the radius CP always forming an angle ψ dependant on the value of x, that is, on the motion of PM, traces out the curve; or, *if the motion of PM be uniform, and CP revolve uniformly, their intersection traces out the curve.*

The ordinates through the points on either side of C corresponding to $x=3r$, $5r$, &c. will be asymptotes. The first branch of the curve will cut the axis X at a point corresponding to $x=2r$, and will extend indefinitely below it. A separate part of the locus will lie between the asymptotes at $3r$ and $5r$, which will cut the axis at the intermediate points $4r$, $6r$, &c. and will extend ad infinitum above and below it: the same on the other side of A. This curve is called the *quadratrix of Dinostratus.*

(14.) Among the various remarkable geometrical properties of this curve, the following are the most important:

1st. Since in the above investigation it appears that we have

$$x=\frac{r}{\frac{1}{2}\pi}\,.\,\psi,$$

we have manifestly

$$\frac{1}{n}x=\frac{r}{\frac{1}{2}\pi}\,.\,\frac{1}{n}\,.\,\psi;$$

if therefore any given abscissa be divided into n equal parts, the corresponding angle ψ, given by the construction of the curve, will also be divided into n equal parts by radii passing through the extremities of the ordinates corresponding to the successive parts of x; *or by means of the quadratrix an angle may be divided into any required number of equal parts.*

(15.) 2dly. By means of this curve (supposing it described) we obtain a solution of the celebrated problem *to find the area of a given circle.*

(Fig. 33.) *AB* being the quadratrix, with radius $r = AC$, describe a circle $A\beta$. Take a position of the radius *CP* near to *CB*, meeting the circle in π, and draw *PK*, $\pi\kappa$, *PM*; then by the nature of the curve

we have $\frac{x}{r} = \frac{\psi}{\frac{1}{2}\pi}$

$$\therefore \frac{\frac{1}{2}\pi}{r} = \frac{\psi}{x} = \frac{\frac{1}{2}\pi - \psi}{r - x} = \frac{\text{arc } \pi\beta = (\text{lim.})\ \pi\kappa}{MC = PK} = (\text{sim. } \triangle)$$

$$\frac{C\kappa = (\text{lim.})\ r}{CK = (\text{lim.})\ CB},$$

$$\text{or } \tfrac{1}{2}\pi \,.\, CB = r^2. \quad \text{Hence } r\pi = \frac{r^3}{\frac{1}{2}CB}.$$

But the area of a circle $= \frac{1}{2}$ radius $\times$ circumference. $\therefore$ with radius r and half circumference π, the area $= r\pi$ which is $\therefore$ given by the quadratrix in terms of *CB*. Hence the appellation of the curve.

For other geometrical properties the student is referred to Robertson's Conic Sections, Appendix, (edit. 1802,) and Leslie's Geometry of Curves.

Another curve of an analogous kind, called the quadratrix of Tschirnhausen, is investigated in Lardner's Algeb. Geometry. See also Peacock's Examples, p. 171, &c.

(16.) The equation $y = \sin. x$ referred to rectangular coordinates, by taking the abscissa x to represent the length of a circular arc to radius 1, and y=its sine, affords one of the most obvious exemplifications of a transcendental locus: it is termed the *sinusoid*, or curve of sines. Similarly we might give geometrical constructions of the equations, $y=\tan. x$, $y=\sec. x$, but for the figures and properties of these curves recourse must be had to the fluxional calculus.

(17.) The case however of $y=\sin. x$ requires a remark. To radius r it becomes $\frac{y}{r}=\sin.\frac{x}{r}$. Also suppose the arc, instead of $\frac{x}{r}$, were $\frac{\pi}{2}-\frac{x}{r}$, the expression would obviously be

$$y=r \cos. \frac{x}{r}.$$

The locus is evidently a curve cutting the axis X at each successive value of $x=\pi$; and during the intervals lying alternately on each side of the axis. The greatest ordinate on each side being that corresponding to $x=\frac{\pi}{2}$, or $y=r$.

Hence it is easy to conceive the curve transferred to a new axis X, parallel to the former, and passing through the lower extremity of the greatest negative ordinate, where it is a tangent to the curve; and the locus will lie wholly above it, touching it at each successive value of $x=2\pi$.

The greatest ordinate is now $=2r$, and the new value of y in any case $=r+$the former y.

We have now therefore $y=r+r \cos. \frac{x}{r}$.

$$\text{or} \quad y-r \cos. \frac{x}{r}-r=0,$$

which is the equation of the companion to the cycloid,

(4. Case 4.) with which this curve is thus evidently identical.

For some curious analogies and properties of this and other classes of curves, see Lardner's Algebraic Geometry, note, p. 474.

(18.) The catenary is another transcendental curve of great importance in mechanics: but not even its construction can be given without the higher calculus.

(19.) The tractrix is a curve, such, that the locus of a *point on the tangent* at a *given distance* from the point of *contact* is a *right line*, and this line is called the directrix of the curve. This curve, and others related to it, can only be investigated fluxionally.

For a full investigation of these and other curves of the same description, the reader is referred to Lardner's Algebraic Geometry, vol. I. and Leslie's Geometry, vol. II.

§. 2.

EXPONENTIAL FUNCTIONS.

(20.) Let there be given the equation involving an exponential function;

$$y=a^x.$$

Here, when $x=0$ $y=1$, and when $x=1$ $y=a$.

If we suppose $a>1$, with $+x$, y increases without limit as x increases.

With $-x$, y diminishes without limit as x increases.

If we suppose $a<1$,

With $+x$, y decreases without limit as x increases.

With $-x$, y increases without limit as x increases.

(21.) Hence the construction of the curve, taking the origin O with rectangular axes, at O, $y=1$. Let that value be represented by y_1. (Fig. 34.) For all values of $+x$, the locus recedes indefinitely from the axis: and on the other side of O approaches it without limit, or the axis becomes an asymptote. If a value of x be taken $=1$, the ordinate at that point $=a$.

Hence, according as a is $>$, or <1, the divergency of the curve will lie on the positive or negative side.

The variable being here involved, as the logarithm of y related to the base a, the locus is called the *logarithmic curve*[a].

If abscissæ are taken with equal differences, the corresponding ordinates are in geometrical progression.

[a] Invented by James Gregory: investigated by Huygens.

POLAR EQUATIONS.

From the nature of a polar equation it appears that it may contain any function, either algebraic or transcendental, of the variable angle or arc which it involves. All the polar equations which we have hitherto considered involve *trigonometrical* functions of θ.

It remains to consider the cases of those polar equations which involve other functions of the variables; as algebraic, or exponential functions.

POLAR EQUATIONS WHICH GIVE SPIRALS.

The loci of polar equations in general are sometimes called *spirals*; but that term is more usually restricted distinctively to the loci of *those polar equations* which involve either *algebraic* or *exponential* functions: in this sense it is here used.

Some writers, instead of polar equations involving r and θ, designate spirals by equations between r and the perpendicular drawn from the pole upon the tangent; but in order to deduce such expressions in a general way from the equations between r and θ, recourse must be had to the differential calculus.

We proceed to investigate a few of the most important examples of spirals.

ALGEBRAIC FUNCTIONS.

(1.) One general class of such equations is represented by the formula

$$r^m = a\theta^{\pm n}.$$

1st. Let n be positive.

Under this case the only examples of any interest are the following:

(2.) Let $n=m=1$: or $r=a\theta$.

In this case when $\theta=1$ $r=a$,
when $\theta=0$ $r=0$,
and $r \propto \theta$.

Hence we have the construction. (Fig. 35.) Taking O as the origin and OX as the fixed axis, assume any value of $r=a$ and an $\angle\ \theta$, which $\therefore =1$.

By the second inference, the locus begins from coinciding with O; and from the third, successive values of r, taken proportional to those of θ, trace out the locus. It is also evident that there is no limit to the successive values of θ. It may be $>\pi$, or $>m\pi$: that is, the locus will form an unlimited number of spiral turns about O.

If θ be taken in arithmetical progression, or the successive radii include equal angles, r also increases by equal differences.

This curve is called the *Spiral of Archimedes*[b].

(3.) Let $m=2$, n being still $=1$ and positive, the equation is $r^2=a\theta$.

[b] Imagined by Conon; investigated by Archimedes.

Here, when $\theta=0$, $r=0$,
and as θ increases indefinitely, r increases indefinitely.

Hence the construction. The locus begins from coincidence with the pole; and if θ be taken increasing by equal differences, r^2 increases by equal differences. Also the number of spiral turns is unlimited.

This curve is called the *parabolic spiral.*

(4.) 2d. Let n be negative, and $m=1$,
and 1st let $n=-1$,
or $r=a\theta^{-1}$, or $r\theta=a$;

that is, $r\theta=$ a circular arc to radius r, $=$ a constant quantity: that is, a circular arc of *constant length,* the angle and radius being variable.

When $r=0$ $\frac{1}{\theta}=0$, or $\theta=\infty$.

When $\theta=0$ $r=\infty$.

Hence the construction. (Fig. 36.) Take any distance OR from the pole, and describe a circular arc RP always$=a$: the point P traces out the locus. From the equation it is evident that successive radii are inversely as the angles they form with the axis. The spiral is supposed to commence from an infinite radius, and never falls into the pole, though in an infinite number of revolutions of the radius it continually approximates to it.

From O take the perpendicular $OB=a$, and draw BS parallel to OR: also the ordinate $PM=y$.

Then we have evidently,

$$y=r.\sin.\theta=\frac{a\sin.\theta}{\theta}$$

$\therefore$ when θ is diminished without limit, we have

$$\frac{\sin.\theta}{\theta}=(\text{lim.})\ 1 \quad \therefore PM=(\text{lim.})\ a;$$

or BS is an asymptote to the spiral. This spiral is called the *hyperbolic.*

(5.) The other conditions remaining, let

$$n=-\tfrac{1}{2}, \text{ or } r=a\theta^{-\frac{1}{2}}$$

$$\therefore\ r^2\theta=a^2, \quad \text{or } r\theta=\frac{a^2}{r}.$$

Here $\theta=0$ gives $r=\infty$

$$r=0 \qquad \frac{1}{\theta^{\frac{1}{2}}}=0, \text{ or } \theta^{\frac{1}{2}}=\infty.$$

Hence the spiral commences from an infinite radius, and continually approaches the pole, but never coincides with it. (Fig. 37.) From O take any distance OR and describe a circular arc $PR=r\theta\ \therefore =\frac{a^2}{r}$, which consequently diminishes as r increases; and when $r=\infty$ and $\therefore \theta=0$, we have $PR=0$, or OR becomes an asymptote.

The area of the circular sector $OPR=\frac{1}{2}r^2\theta=\frac{1}{2}a^2=$ a constant area. This spiral is called *the Lituus* [b].

The student will not fail to notice the analogy between these classes of spirals and the parabolic and hyperbolic curves from which their names are derived.

(6.) Let the equation be

$$r=\frac{a\theta^2}{\theta^2-1}$$

$$\therefore\ r\theta^2-r=a\theta^2, \text{ or } \theta^2=\frac{r}{r-a}.$$

[b] This spiral and the last were the inventions of Cotes.

Hence, if $\theta = 1$ $r = \infty$

if $r = a$ $\theta = \infty$

if $r < a$, $-\theta^2$, impossible.

(Fig. 40.) Hence the construction. Describe a circle with radius $= a$: no part of the locus lies within it: it extends round it, and the *circumference of the circle* is an *asymptote* to the spiral.

Take an angle aOS, giving an arc $\theta =$ length of radius 1, then $r = \infty$, or the locus never meets OS produced indefinitely.

There is, however, nothing thus far to shew that the curve *approaches* OS, and in fact it is not an asymptote. The actual position of the asymptote is found fluxionally: it is a line parallel to OS, between it and the curve.

SPIRALS.

EXPONENTIAL FUNCTIONS.

(7.) Let the equation be $r = a^\theta$.

Here, when $\theta = 0$ $r = a^0 = 1$, the quantity taken as the unit must always be finite, or r never can $= 0$.

As θ increases without limit, r increases without limit.

$\theta = \log. r$ to base a.

If θ increase in arithmetical progression, r increases in geometrical.

Hence the construction. (Fig. 38.) Since r is never $= 0$, the locus never falls into the pole.

If about O radii including equal angles be drawn, the radii will be in continued proportion, and the locus will extend to an infinite number of turns. It is called the *logarithmic spiral*[c].

(8.) If radii including equal angles be taken indefinitely near to each other, the portions of the curve intercepted will in the limit be rectilinear. Hence we have triangles having one angle equal, and the sides about it proportionals; the Δs are consequently equiangular; or *the radii form equal angles with the curve.* Hence it is sometimes called the *equiangular spiral.*

(9.) From this geometrical property is derived an easy mechanical construction.

(Fig. 39.) A line of indefinite length, PY, is capable of moving in any direction on a plane, but always passing through the fixed centre O. At its extremity P it is bent at any angle, and the bent part is the axis to a small wheel (w): an impulse is given to the wheel, which, by the natural course of its motion, (always at right angles to the bent part,) traces out the equiangular spiral.

If the angle at $P=0$, or the line, be not bent, the curve described is obviously *a circle;* which is a geometrical species of the equiangular spiral. It is also an algebraical species, corresponding to the condition in the equation $a=1$.

For a variety of examples of other spirals see Jephson's Fluxional Calculus, chap. 10; also Peacock's Examples, p. 179, &c.

---

[c] This curve seems to have been invented by Des Cartes, and was fully investigated by James Bernoulli.

ON SIMILAR CURVES AND SPIRALS.

(1.) In the most general forms of the equations to curves, several constant quantities enter, which determine the *position* of the locus. When, however, we take the most simple conditions, such as the axes rectangular, and the origin at the vertex or centre, the constant quantities which remain are now only such as determine the *magnitude* of the locus in certain given directions: that is, for curves of the same degree and denomination, the varieties in magnitude and form of which they are susceptible depend upon the values of these constant quantities, and the ratio subsisting between them. These constants are sometimes called *parameters.*

(2.) Def. Two curves of the same kind are said to be *similar* when any *abscissæ* being taken in the two curves, in a *given ratio* to each other, the corresponding *ordinates* are in the *same ratio.*

To find under what conditions this will take place, we have only to observe that the equations being of the same form, the ordinate in both curves is the same function of the abscissa. And further, the equations being homogeneous, (V. 4.) let the constant quantities (if more than one) be taken in the same ratio in the two curves: it may then easily be shewn that if in the two curves we take abscissæ in any given ratio, as that of the constant quantities, the corresponding ordinates being the same functions of these abscissæ, and involved with constant quantities which are in the same ratio, will themselves be also in the same ratio.

This will readily appear from considering a very few cases.

(3.) In the equation to the conic sections

$$y^2 = px + qx^2 \qquad \text{where } q = \frac{p}{2a}$$

the constants which determine the species and magnitude of each curve are p and a. In two curves, therefore, let them be assumed

$$\frac{p}{p_2} = \frac{a}{a_2} \qquad \text{whence } q = q_2.$$

Also take $\dfrac{x}{x_2} = \dfrac{p}{p_2}$

$$\therefore \quad \frac{x^2}{x_2^2} = \frac{px}{p_2x_2} = \frac{qx^2}{q_2x_2^2} = \frac{px + qx^2}{p_2x_2 + q_2x_2^2} = \frac{y}{y_2^2}:$$

whence the curves are similar by DEF.

This assumption also obviously gives the condition in the ellipse and hyperbola, that the axes are in the same ratio. *Ellipses and hyperbolas are therefore similar when the ratios of their axes are the same.*

For the parabola, by substituting $q = 0$, the same investigation applies. In this case the ratio

$$\frac{x^2}{x_2^2} = \frac{px}{p_2x_2} = \frac{y^2}{y_2^2}$$

is unrestricted to any values of p and p_2, or *all parabolas are similar.*

(4.) The application of a similar mode of investigation to other simple curves is so obvious as to render particular statements of it unnecessary. We may easily shew, by exactly similar steps, that all parabolæ, and rectangular hyperbolæ, of any the same order, are similar curves.

In the same way, from the equations of the cissoid, and of the lemniscata, which each contain but one constant, we may deduce that all curves of each of these kinds are similar.

The equation to the conchoid contains two constants: and by pursuing the same mode of investigation we find that two conchoids are similar if the ratio between the modulus and the distance from the node to the base in each curve is the same.

(5.) The same principles apply also to transcendental curves. Thus if we take the general equation to the cycloidal curves, we have

$$y=r\ (1+\cos.\ \psi)$$
$$x=r\ (m\psi+n\ \sin.\ \psi.)$$

In two curves let $m=m_2$ and $n=n_2$:

$$\text{also take } \frac{x}{x_2}=\frac{r}{r_2}$$

$$\therefore\ \frac{x}{x_2}=\frac{r(m\psi+n\ \sin.\ \psi)}{r_2\ (m\psi_2+n\ \sin.\ \psi_2)}$$

$$\therefore\ m\psi+n\ \sin.\ \psi=m\psi_2+n\ \sin.\ \psi_2$$

$$\therefore\ m\psi=m\psi_2,\ \text{and}\ n\ \sin.\ \psi=n\ \sin.\ \psi_2:$$

hence $\psi=\psi_2$, and $\cos.\ \psi=\cos.\ \psi_2$.

Whence we have,

$$\frac{y}{y_2}=\frac{r\ (1+\cos.\ \psi)}{r_2(1+\cos.\ \psi)}=\frac{r}{r_2}\ \therefore=\frac{x}{x_2}$$

or any cycloidal curves are similar, in which the ratios between the radii of the revolving and the generating circles are the same: and the quantity n is the same;

that is, equal to unity in the cycloids, or to nothing in the companion.

Hence all common cycloids are similar, since in all these $m = 1$.

For the further fluxional investigation of similar curves, the student is referred to Jephson's Fluxional Calculus, p. 264.

(6.) Def. Two spirals of the same kind are said to be similar when at equal values of the variable angle the radii are proportionals.

Thus with the same angle θ in two spirals, let the radii be r, ρ; and with another angle θ_2 let them be $r_2\,\rho_2$. Then the spirals are similar if $\frac{r}{r_2} = \frac{\rho}{\rho_2}$.

(7.) With the spiral equation $r^m = a\theta^n$ we have,

$$\frac{r^m}{\rho^m} = \frac{a\theta^n}{\alpha\theta^n} \quad \frac{r_2^{\,m}}{\rho_2^{\,m}} = \frac{a\theta_2^{\,n}}{\alpha\theta_2^{\,n}} \quad \therefore \frac{r}{\rho} = \frac{r_2}{\rho_2}$$

or all spirals in each of the kinds included under this equation are similar.

(8.) With the logarithmic spiral, whose equation is

$$r = a^\theta$$

$$\text{we have } \frac{r}{\rho} = \frac{a^\theta}{\alpha^\theta},\ \text{and } \frac{r_2}{\rho_2} = \frac{a^\theta{}_2}{\alpha^\theta{}_2},$$

$$\text{whence } \frac{r^{\frac{1}{\theta}}}{\rho^{\frac{1}{\theta}}} = \frac{r_2^{\frac{1}{\theta_2}}}{\rho_2^{\frac{1}{\theta_2}}} = \frac{a}{\alpha}$$

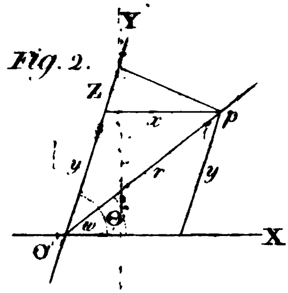
$$\text{or } \frac{r^{\frac{\theta_2}{\theta}}}{\rho^{\frac{\theta_2}{\theta}}} = \frac{r_2}{\rho_2}\,,$$

but this can only $= \frac{r}{\rho}$ when we have $r = \rho$, and $\therefore a = \alpha$, or logarithmic spirals can only be similar when they are equal.

Upon geometrical principles also it appears, from Spir. (8), that logarithmic spirals will be similar when the angle which the radius forms with the curve, or with its tangent, is the same in each; but from the triangles thus formed it is evident that when this angle is equal in two spirals the spirals are also equal.

THE END.

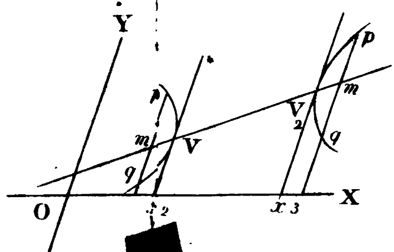
Fig. 2.



Fi

O

Fig. 5.



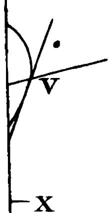


Fig. 7.

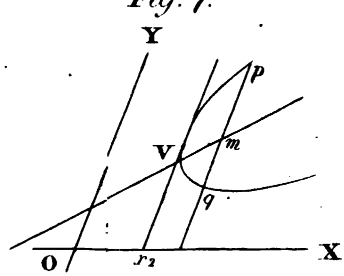


PLATE I.

Fig. 2.

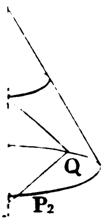


Fig. 5.

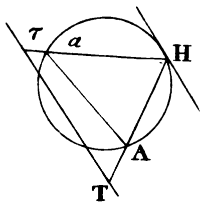


Fig. 4.

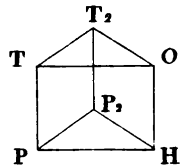


Fig. 8.

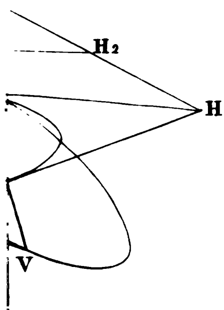
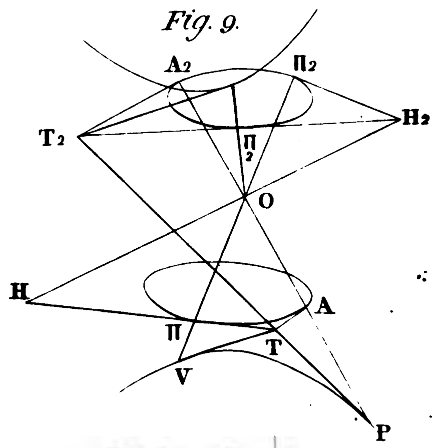


Fig. 9.



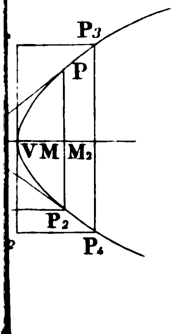


Fig. 3.

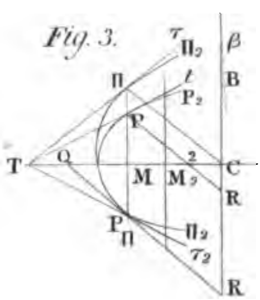



Fig
2

Fig. 14.

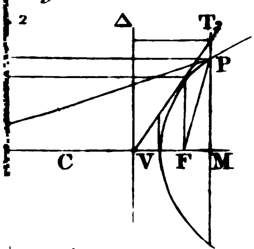


Fig. 11.

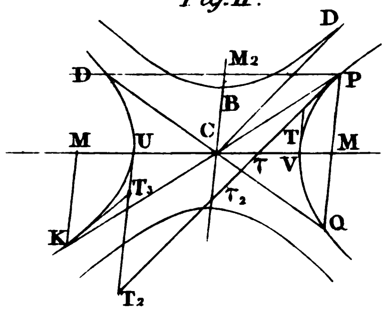


Fig. 15.

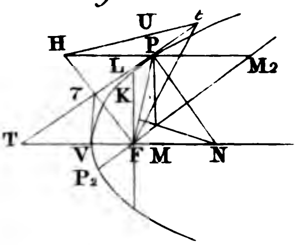
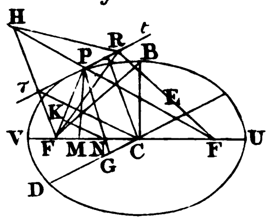


Fig. 16.



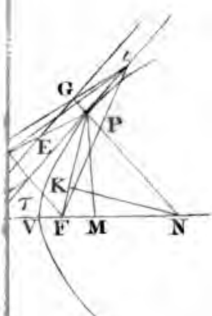
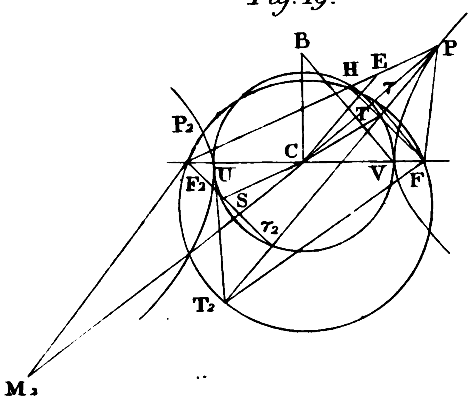
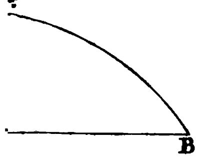


Fig. 19.





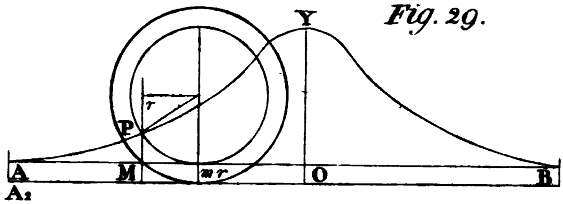
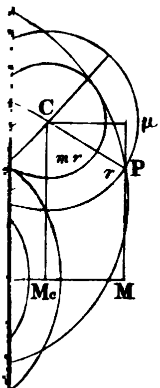


Fig. 29.



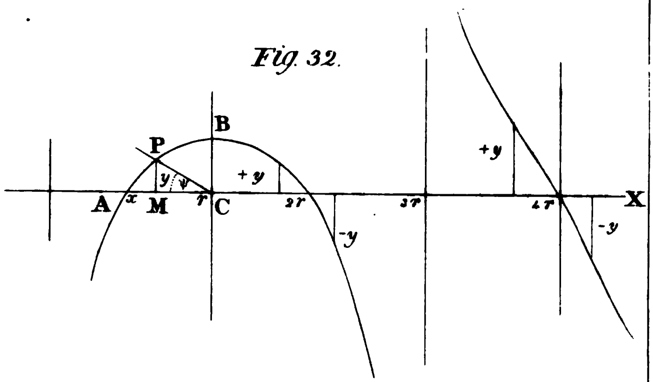


Fig. 32.

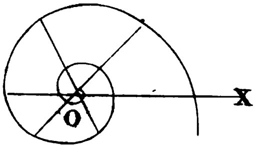


Fig. 35.

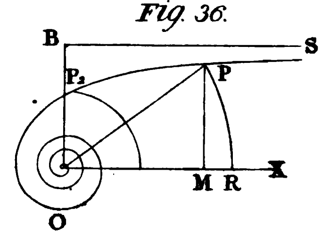
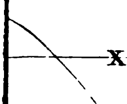


Fig. 36.



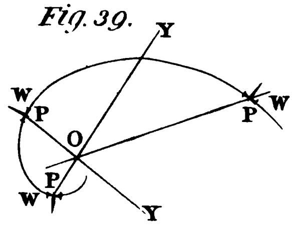


Fig. 39.

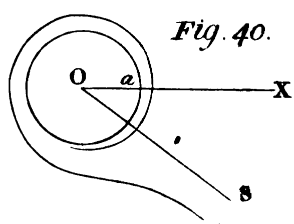


Fig. 40.